Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021

Solutions to Assignment #2
Solving Equations
Due on Friday, 5 February.

The following problem was posed by the Indian mathematician Bhaskara (*c.* 1114–1185 A.D.) in a book dedicated to his daughter Lilavati:

> Seven times half the square root of a flock of geese was observed to march slowly away and two were seen fighting playfully in the water. Say, what was the number of geese?

For those interested in the history of mathematics, Bhaskara developed a number of techniques that anticipated portions of both differential and integral calculus.

1. Restate the problem given above as an equation. *[1]*

SOLUTION. Let x be the total number of geese in the flock. The problem gives a breakdown of the flock into two parts: the "seven times half the square root of a flock of geese" that marched away and the two that "were seen fighting playfully in the water". This gives us the following equation:

$$\frac{7}{2}\sqrt{x} + 2 = x \qquad \blacksquare$$

2. Solve the equation you obtained in **1** by hand. *[1]*

SOLUTION. We isolate the $\sqrt{x}$ and then square both sides to obtain a quadratic equation, which we will solve using the quadratic formula. Note that x must be positive.

$$\begin{aligned}\frac{7}{2}\sqrt{x} + 2 = x \implies \frac{7}{2}\sqrt{x} = x - 2 \implies \sqrt{x} = \frac{2}{7}(x-2) \implies x = \frac{4}{49}\left(x^2 - 4x + 4\right) \\ \implies x = \frac{4}{49}x^2 - \frac{16}{49}x + \frac{16}{49} \implies \frac{4}{49}x^2 - \frac{16}{49}x - \frac{49}{49}x + \frac{16}{49} = 0 \\ \implies \frac{4}{49}x^2 - \frac{65}{49}x + \frac{16}{49} = 0 \implies 4x^2 - 65x + 16 = 0\end{aligned}$$

We now throw the quadratic formula at the last equation.

$$\begin{aligned}4x^2 - 65x + 16 = 0 \implies x &= \frac{-(-65) \pm \sqrt{(-65)^2 - 4 \cdot 4 \cdot 16}}{2 \cdot 4} \\ &= \frac{65 \pm \sqrt{4225 - 256}}{8} = \frac{65 \pm \sqrt{3969}}{8} \\ &= \frac{65 \pm 63}{8} = \frac{128}{8} \text{ or } \frac{2}{8} = 16 \text{ or } \frac{1}{4}\end{aligned}$$

Since a "flock of geese" consisting of a quarter of a goose makes little sense, the flock had sixteen geese. $\blacksquare$

3. Solve the equation you obtained in **1** using `Maple`. *[1]*

Note: The basic tool you will need to do **3** is `Maple`'s `solve` command, which has many options and variations. Make sure to ask for help if you need it!

SOLUTION. Here is what `Maple` gave:

> $solve\left(\left(\frac{7}{2}\right)\cdot \mathrm{sqrt}(x)+2=x,x\right)$

$$16 \qquad (1)$$

Observe that `Maple` only gave the integer solution, probably because the fractional solution to the corresponding quadratic equation does not actually satisfy the original equation, which is what `Maple` was asked to solve. ■

The *hyperbolic functions* include:

$$\sinh(x)=\frac{e^x-e^{-x}}{2} \qquad \cosh(x)=\frac{e^x+e^{-x}}{2} \qquad \tanh(x)=\frac{\sinh(x)}{\cosh(x)}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

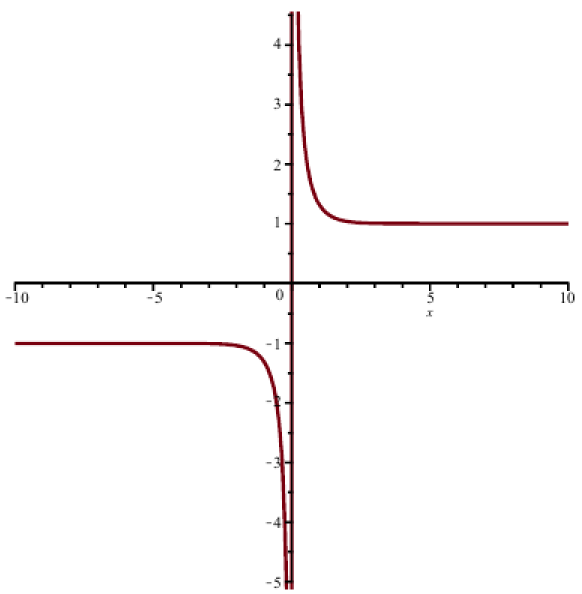
$$\operatorname{csch}(x)=\frac{1}{\sinh(x)} \qquad \operatorname{sech}(x)=\frac{1}{\cosh(x)} \qquad \coth(x)=\frac{\cosh(x)}{\sinh(x)}=\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

The names of these function are usually pronounced something like "sinch", "kosh", "tanch", "co-seech", "seech", and "kotch", respectively. Your main task in the rest of this assignment will be to investigate $\coth(x)$ and find its inverse.

4. Use `Maple` to graph $\coth(x)$. *[1]*

SOLUTION. Here we are:

> $plot(\coth(x))$



Note the vertical asymptotes at $x=0$ and the horizontal asymptotes at $y=\pm 1$. ■

5. Use `Maple`'s ability to solve equations symbolically to find an expression for $\operatorname{arccoth}(x)$, the inverse function of $\coth(x)$. *[3]*

SOLUTION. Note that $y = \operatorname{arccoth}(x) \iff x = \coth(y)$. Asking `Maple` to solve $x = \coth(y)$ for y gave something that is correct,

```
> solve(x = coth(y), y)
```

$$\operatorname{arccoth}(x) \tag{2}$$

but useless for this assignment. Asking it to solve $x = \dfrac{e^y + e^{-y}}{e^y - e^{-y}}$ for y gave something more useful:

```
> solve(x = (exp(y) + exp(-y))/(exp(y) - exp(-y)), y)
```

$$\ln\left(\frac{\sqrt{(x-1)\,(1+x)}}{x-1}\right), \ln\left(-\frac{\sqrt{(x-1)\,(1+x)}}{x-1}\right) \tag{3}$$

Note that because of the properties of square roots and logarithms, the solutions given by `Maple` could be rewritten in many different ways. ■

6. Work out an expression for $\operatorname{arccoth}(x)$ yourself. (If this is different from what `Maple` gave you in **5**, you may well be correct, but try to explain, if you can, why they amount to the same thing.) *[3]*

SOLUTION. We will solve $x = \dfrac{e^y + e^{-y}}{e^y - e^{-y}}$ for y by first solving for e^y and then applying the natural logarithm function.

$$\begin{aligned}
x = \frac{e^y + e^{-y}}{e^y - e^{-y}} &\implies x\left(e^y - e^{-y}\right) = e^y + e^{-y} \implies xe^y - xe^{-y} = e^y + e^{-y} \\
&\implies \left(xe^y - x\frac{1}{e^y}\right)e^y = \left(e^y + \frac{1}{e^y}\right)e^y \implies x\left(e^y\right)^2 - x = \left(e^y\right)^2 + 1 \\
&\implies (x-1)\left(e^y\right)^2 = x+1 \implies \left(e^y\right)^2 = \frac{x+1}{x-1} \\
&\implies e^y = \pm\sqrt{\frac{x+1}{x-1}}
\end{aligned}$$

Since $e^y > 0$ for all real number values of y, we must have $e^y = \sqrt{\dfrac{x+1}{x-1}}$, from which it follows that $y = \ln\left(\sqrt{\dfrac{x+1}{x-1}}\right)$. Note that this answer is the same as the first one given by `Maple`, because $\sqrt{\dfrac{x+1}{x-1}} = \sqrt{x+1}\cdot\dfrac{1}{\sqrt{x-1}} = \sqrt{x+1}\cdot\dfrac{\sqrt{x-1}}{x-1} = \dfrac{\sqrt{(x+1)(x-1)}}{x-1}$. `Maple`'s second answer is the one we discarded because e^y must be positive for real values of y. ■