Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021

Solution to Quiz #7
Tuesday, 9 March.

Show all your work! Simplify where you conveniently can.

1. Find the domain and any and all intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, maximum and minimum points, intervals of curvature, and inflection points of $f(x) = \left(\frac{x}{x+1}\right)^2$. Sketch the graph of $f(x)$ based on this information. *[5]*

SOLUTION. *i. Domain.* $f(x) = \left(\frac{x}{x+1}\right)^2$ is defined for all x except for $x = -1$, which makes the denominator 0, so its domain is $(-\infty, -1) \cup (-1, \infty) = \{ x \in \mathbb{R} \mid x \neq -1 \}$.

ii. Intercepts. For the y-intercept, we set $x = 0$, then $y = f(0) = \left(\frac{0}{0+1}\right)^2 = 0$. For the x-intercept, we set $y = 0$ and solve for x: $y = \left(\frac{x}{x+1}\right)^2 = 0$ can only happen when the numerator of the fraction is 0, *i.e.* when $x = 0$.

iii. Vertical asymptotes. Since $f(x)$ is a composition of continuous functions, it is continuous wherever it is defined, *i.e.* for all $x \neq 1$. We check to see if there are any vertical asymptotes at this point:

$$\lim_{x \to -1^-} \left(\frac{x}{x+1}\right)^2 = \lim_{x \to -1^-} \frac{x^2}{(x+1)^2} \begin{matrix} \to 1 \\ \to 0^+ \end{matrix} = +\infty$$

$$\lim_{x \to -1^+} \left(\frac{x}{x+1}\right)^2 = \lim_{x \to -1^+} \frac{x^2}{(x+1)^2} \begin{matrix} \to 1 \\ \to 0^+ \end{matrix} = +\infty$$

Thus $f(x) = \left(\frac{x}{x+1}\right)^2$ has a vertical asymptote at $x = -1$, heading off to $+\infty$ as x approaches -1 from either side.

iv. Horizontal asymptotes. We check to see what $f(x)$ does as $x \to \pm\infty$:

$$\begin{aligned} \lim_{x \to -\infty} \left(\frac{x}{x+1}\right)^2 &= \lim_{x \to -\infty} \frac{x^2}{(x+1)^2} \begin{matrix} \to +\infty \\ \to +\infty \end{matrix} \quad \text{so we can apply l'Hôpital's Rule} \\ &= \lim_{x \to -\infty} \frac{\frac{d}{dx} x^2}{\frac{d}{dx}(x+1)^2} = \lim_{x \to -\infty} \frac{2x}{2(x+1)} \begin{matrix} \to -\infty \\ \to -\infty \end{matrix} \quad \text{so} \ldots \\ &= \lim_{x \to -\infty} \frac{\frac{d}{dx} 2x}{\frac{d}{dx} 2(x+1)} = \lim_{x \to -\infty} \frac{2}{2} = 1 \end{aligned}$$

and

$$\lim_{x\to+\infty}\left(\frac{x}{x+1}\right)^2 = \lim_{x\to+\infty}\frac{x^2}{(x+1)^2}\begin{array}{l}\to+\infty\\ \to+\infty\end{array} \quad \text{so we can apply l'Hôpital's Rule}$$
$$= \lim_{x\to+\infty}\frac{\frac{d}{dx}x^2}{\frac{d}{dx}(x+1)^2} = \lim_{x\to+\infty}\frac{2x}{2(x+1)}\begin{array}{l}\to+\infty\\ \to+\infty\end{array} \quad \text{so} \ldots$$
$$= \lim_{x\to+\infty}\frac{\frac{d}{dx}2x}{\frac{d}{dx}2(x+1)} = \lim_{x\to+\infty}\frac{2}{2} = 1\,,$$

so $f(x) = \left(\frac{x}{x+1}\right)^2$ has a horizontal asymptote of $y=1$ in both directions.

v. Intervals of increase and decrease and maxima and minima. We need to compute the first derivative:

$$f'(x) = \frac{d}{dx}\left(\frac{x}{x+1}\right)^2 = \frac{d}{dx}\frac{x^2}{(x+1)^2} = \frac{\frac{d}{dx}\left[x^2\right](x+1)^2 - x^2\left[\frac{d}{dx}(x+1)^2\right]}{\left((x+1)^2\right)^2}$$
$$= \frac{2x(x+1)^2 - x^2 2(x+1}{(x+1)^4} = \frac{2x(x+1)-2x^2}{(x+1)^3} = \frac{2x^2+2x-2x^2}{(x+1)^3} = \frac{2x}{(x+1)^3}$$

It follows that $f'(x)$ is undefined at $x=-1$, just like $f(x)$, and $=0$ exactly when the numerator $=0$, *i.e.* exactly when $x=0$. Observe that when $x<-1$, both the numerator and the denominator of $f'(x) = \dfrac{2x}{(x+1)^3}$ are negative, so $f'(x)>0$ and $f(x)$ is increasing; when $-1<x<0$, the numerator is negative and the denominator is positive, so $f'(x)<0$ and $f(x)$ is decreasing, and when $x>0$, the numerator and denominator are both positive, so $f'(x)>0$ and $f(x)$ is increasing. This tells us that the critical point at $x=0$ is a minimum. We summarize all this in the following table:

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,+\infty)$
$f'(x)$	$+$	undef	$-$	0	$+$
$f(x)$	$\uparrow$	undef	$\downarrow$	min	$\uparrow$

Thus $f(x)$ is increasing on the intervals $(-\infty,-1)$ and $(0,+\infty)$ and decreasing on the interval $(-1,0)$, and it has a minimum at $x=0$. Note also that $f(0)=0$.

vi. Curvature and points of inflection. We need to compute the second derivative:

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\frac{2x}{(x+1)^3} = \frac{\left[\frac{d}{dx}2x\right](x+1)^3 - 2x\left[\frac{d}{dx}(x+1)^3\right]}{\left((x+1)^3\right)^2}$$
$$= \frac{2(x+1)^3 - 2x3(x+1)^2}{(x+1)^6} = \frac{2(x+1)-6x}{(x+1)^4} = \frac{2x+2-6x}{(x+1)^4} = \frac{2-4x}{(x+1)^4}$$

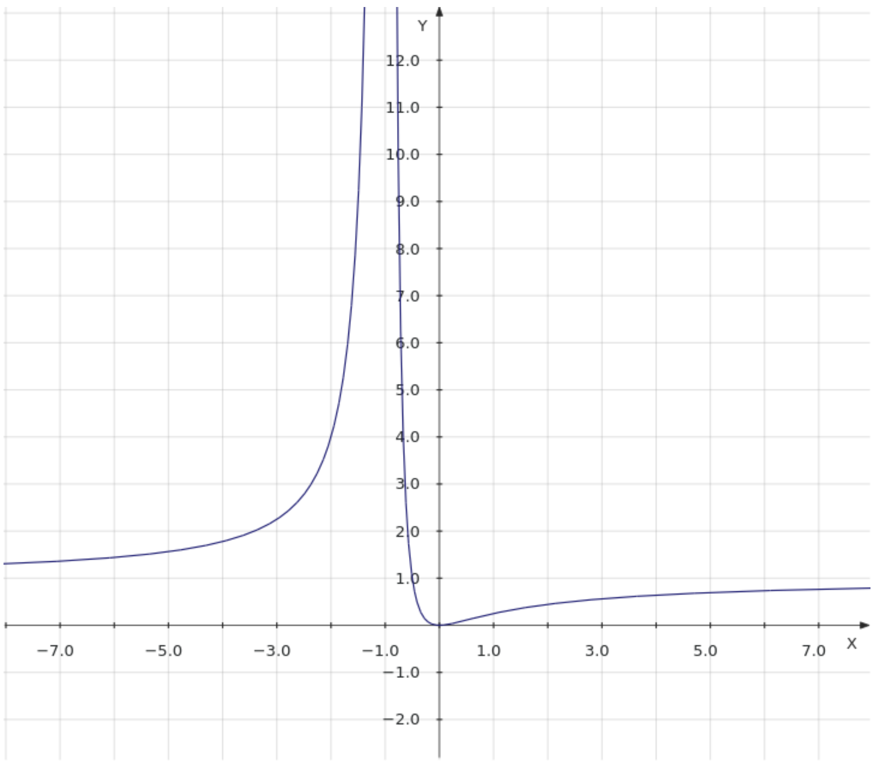
It follows that $f''(x)$ is undefined at $x=-1$, just like $f(x)$ and $f'(x)$, and $=0$ exactly when the numerator $=0$, *i.e.* exactly when $x=\frac{1}{2}$. Observe that when $x<-1$, the numerator of

$f''(x)$ is positive and the denominator is positive, so $f''(x)$ is positive and $f(x)$ is concave up; when $-1 < x < \frac{1}{2}$, the numerator is positive and the denominator is positive, so $f''(x)$ is positive and $f(x)$ is concave up; and when $x > \frac{1}{2}$, the numerator is negative and the denominator is positive, so $f''(x)$ is negative and $f(x)$ is concave down. This tells us that $x = \frac{1}{2}$ is an inflection point. We summarize all this in the following table:

x	$(-\infty, -1)$	-1	$\left(-1, \frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2}, +\infty\right)$
$f''(x)$	$+$	undef	$+$	0	$-$
$f(x)$	$\smile$	undef	$\smile$	inflect	$\frown$

Thus $f(x)$ is concave up on $(-\infty, -1)$ and $\left(-1, \frac{1}{2}\right)$ and concave down on $\left(\frac{1}{2}, +\infty\right)$, and it has an inflection point at $x = \frac{1}{2}$. Note also that $f\left(\frac{1}{2}\right) = \frac{1}{9}$.

vii. The sketch. Cheating a ~~little~~ ~~lot~~ somewhat, here is a graph of $f(x) = \left(\dfrac{x}{x+1}\right)^2$, as drawn by a program called `kmplot`.



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