Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals

TRENT UNIVERSITY, Winter 2021

Quiz #6

Tuesday, 2 March.

Show all your work! Simplify where you conveniently can.

1. Find the the maximum and minimum values, if any, of $f(x) = \dfrac{x^2}{e^{x^2}} = x^2 e^{-x^2}$ on its domain. *[5]*

SOLUTION. Since $e^t > 0$ for all t, $f(x) = \dfrac{x^2}{e^{x^2}}$ is defined for all x; as the function is a composition of continuous functions and is defined everywhere, it is also continuous for all x. This means that there are no potential vertical asymptotes to consider.

To find the critical points, we need to compute the derivative:

$$f'(x) = \frac{\left[\frac{d}{dx}x^2\right]e^{x^2} - x^2\left[\frac{d}{dx}e^{-x^2}\right]}{\left(e^{x^2}\right)^2} = \frac{2xe^{x^2} - x^2e^{x^2}\frac{d}{dx}\left(x^2\right)}{\left(e^{x^2}\right)^2}$$
$$= \frac{2xe^{x^2}\left(1-x^2\right)}{\left(e^{x^2}\right)^2} = \frac{2x\left(1-x^2\right)}{e^{x^2}}$$

Since $e^{x^2} > 0$ for all x, $f'(x)$ is defined for all x. $f'(x) = 0$ exactly when $2x\left(1-x^2\right) = 2x(1+x)(1-x) = 0$, *i.e.* exactly when $x = 0$, $x = -1$, or $x = 1$. We plug these values into our original function to see what values we get: $f(0) = 0^2e^{-0^2} = 0 \cdot 1 = 0$, $f(-1) = (-1)^2e^{-(-1)^2} = 1 \cdot e^-1 = e^{-1} = \frac{1}{e}$, and $f(-1) = 1^2e^{-1^2} = 1 \cdot e^-1 = e^{-1} = \frac{1}{e}$.

It remains to compare these values with what the function does at the ends of its domain of $(-\infty, \infty)$. Note that $\lim_{x\to-\infty} x^2 = +\infty$, $\lim_{x\to+\infty} x^2 = +\infty$, and $\lim_{t\to+\infty} e^t = +\infty$. It follows that, with a little help from l'Hôpital's Rule:

$$\lim_{x\to-\infty} \frac{x^2}{e^{x^2}} \begin{matrix}\to +\infty \\ \to +\infty\end{matrix} = \lim_{x\to-\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^{x^2}} = \lim_{x\to-\infty} \frac{2x}{2xe^{x^2}} = \lim_{x\to-\infty} \frac{1}{e^{x^2}} \begin{matrix}\to 1 \\ \to +\infty\end{matrix} = 0^+$$
$$\lim_{x\to+\infty} \frac{x^2}{e^{x^2}} \begin{matrix}\to +\infty \\ \to +\infty\end{matrix} = \lim_{x\to+\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^{x^2}} = \lim_{x\to+\infty} \frac{2x}{2xe^{x^2}} = \lim_{x\to+\infty} \frac{1}{e^{x^2}} \begin{matrix}\to 1 \\ \to +\infty\end{matrix} = 0^+$$

Putting all of the above together, $f(x) = \dfrac{x^2}{e^{x^2}} = x^2e^{-x^2}$ has a maximum value of $\dfrac{1}{e} = e^{-1}$, which it achieves at $x = \pm 1$, and a minimum value of $x = 0$, which it achieves at $x = 0$. ■