Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021
Solutions to the Monstrous **Take-Home Final Examination** of Horror

Available on Blackboard from ~~12:00~~ 6:30 a.m. on Monday, 19 April.
Due on Blackboard by ~~11:59 p.m. on Wednesday, 21~~ 6:30 a.m. on Thursday, 22 April.

Submission: Scans or photos of handwritten work are entirely acceptable so long as they are legible and in some common format; solutions submitted as a single pdf are strongly preferred. If submission via Blackboard's Assignments module fails repeatedly, then (only as a *last* resort) email them to the instructor at: `sbilaniuk@trentu.ca`

Allowed aids: For this exam, you are permitted to use your textbook and all other course material, from this and any other mathematics course(s) you have taken or are taking now, but *you may not use any other sources or aids, nor give or receive any help*, except to ask the instructor to clarify questions and to use a calculator (any that you like).

Instructions: Do parts **X** and **Y**, and, if you wish, part **Z**. Please show all your work and justify all your answers. *If in doubt about something,* **ask!**

Part Xenomorph. Do *all four* (4) of **1**–**4**. *[Subtotal = 72]*

1. Compute $\frac{dy}{dx}$ as best you can in any *five* (5) of **a**–**f**. *[20 = 5 × 4 each]*

a. $y = \sqrt{\frac{x-1}{x+1}}$ **b.** $y = \int_0^{x^2} \sin(t)\,dt$ **c.** $y = \arctan\left(e^{2x}\right)$

d. $e^{xy} = x$ **e.** $y = \ln\left(x^2-1\right)$ **f.** $y = \frac{x^2-1}{x^4-1}$

SOLUTIONS. **a.** *Power, Chain, and Quotient Rules.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\sqrt{\frac{x-1}{x+1}} = \frac{d}{dx}\left(\frac{x-1}{x+1}\right)^{1/2} = \frac{1}{2}\left(\frac{x-1}{x+1}\right)^{-1/2} \cdot \frac{d}{dx}\left(\frac{x-1}{x+1}\right)\\
&= \frac{1}{2}\left(\frac{x-1}{x+1}\right)^{-1/2} \cdot \frac{\left[\frac{d}{dx}(x-1)\right](x+1) - (x-1)\left[\frac{d}{dx}(x+1)\right]}{(x+1)^2}\\
&= \frac{1}{2}\left(\frac{x+1}{x-1}\right)^{1/2} \cdot \frac{1\cdot(x+1)-(x-1)\cdot 1}{(x+1)^2} = \frac{1}{2}\left(\frac{x+1}{x-1}\right)^{1/2} \cdot \frac{x+1-x+1}{(x+1)^2}\\
&= \frac{1}{2}\left(\frac{x+1}{x-1}\right)^{1/2} \cdot \frac{2}{(x+1)^2} = \frac{1}{(x+1)^{3/2}(x-1)^{1/2}} \qquad \square\end{aligned}$$

b. *Fundamental Theorem of Calculus, Chain Rule, and Power Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^{x^2} \sin(t)\,dt = \sin\left(x^2\right) \cdot \frac{d}{dx}x^2 = \sin\left(x^2\right) \cdot 2x = 2x\sin\left(x^2\right) \qquad \square$$

b. *Integration, Chain Rule, and Power Rule.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_0^{x^2}\sin(t)\,dt = \frac{d}{dx}\left(-\cos(t)\big|_0^{x^2}\right) = \frac{d}{dx}\left(\left[-\cos\left(x^2\right)\right]-\left[-\cos(0)\right]\right)\\ &= \frac{d}{dx}\left(-\cos\left(x^2\right)-[-1]\right) = \frac{d}{dx}\left(1-\cos\left(x^2\right)\right) = \left[\frac{d}{dx}1\right]-\left[\frac{d}{dx}\cos\left(x^2\right)\right]\\ &= 0-\left[-\sin\left(x^2\right)\cdot\frac{d}{dx}x^2\right] = \sin\left(x^2\right)\cdot 2x = 2x\sin\left(x^2\right) \qquad \square\end{aligned}$$

c. *Chain Rule, and lots of it.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\arctan\left(e^{2x}\right) = \frac{1}{1+\left(e^{2x}\right)^2}\cdot\frac{d}{dx}e^2x = \frac{1}{1+e^{4x}}\cdot e^{2x}\cdot\frac{d}{dx}(2x)\\ &= \frac{1}{1+e^{4x}}\cdot e^{2x}\cdot 2 = \frac{2e^{2x}}{1+e^{4x}} \qquad \square\end{aligned}$$

d. *Implicit differentiation, Chain and Product Rules, and algebra.*

$$\begin{aligned}e^{xy}=x &\implies \frac{d}{dx}e^{xy} = \frac{d}{dx}x \implies e^{xy}\cdot\frac{d}{dx}(xy)=1 \implies e^{xy}\cdot\left(\left[\frac{d}{dx}x\right]y+x\left[\frac{d}{dx}y\right]\right)=1\\ &\implies 1\cdot y+x\cdot\frac{dy}{dx}=e^{-xy} \implies x\cdot\frac{dy}{dx}=e^{-xy}-y \implies \frac{dy}{dx}=\frac{e^{-xy}-y}{x} \qquad \square\end{aligned}$$

d. *Algebra and Quotient Rule.*

$$\begin{aligned}e^{xy}=x &\implies xy=\ln(x) \implies y=\frac{\ln(x)}{x}\\ &\implies \frac{dy}{dx}=\frac{d}{dx}\left(\frac{\ln(x)}{x}\right)=\frac{\left[\frac{d}{dx}\ln(x)\right]x-\ln(x)\left[\frac{d}{dx}x\right]}{x^2}\\ &= \frac{\frac{1}{x}\cdot x-\ln(x)\cdot 1}{x^2}=\frac{1-\ln(x)}{x^2} \qquad \square\end{aligned}$$

e. *Chain Rule and Power Rule.*

$$\frac{dy}{dx}=\frac{d}{dx}\ln\left(x^2-1\right)=\frac{1}{x^2-1}\cdot\frac{d}{dx}\left(x^2-1\right)=\frac{1}{x^2-1}\cdot(2x-0)=\frac{2x}{x^2-1} \qquad \square$$

f. *Quotient Rule, Power Rule, and a little algebra.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2-1}{x^4-1}\right)=\frac{\left[\frac{d}{dx}\left(x^2-1\right)\right]\left(x^4-1\right)-\left(x^2-1\right)\left[\frac{d}{dx}\left(x^4-1\right)\right]}{\left(x^4-1\right)^2}\\ &= \frac{2x\left(x^4-1\right)-\left(x^2-1\right)\cdot 4x^3}{\left(x^4-1\right)^4}=\frac{2x^5-2x-4x^5+4x^3}{\left(x^4-1\right)^2}=\frac{-2x^5+4x^3-2x}{\left(x^4-1\right)^2}\\ &= \frac{-2x\left(x^4-2x^2+1\right)}{\left(x^4-1\right)^2}=\frac{-2x\left(x^2-1\right)^2}{\left(x^2-1\right)^2\left(x^2+1\right)^2}=\frac{-2x}{\left(x^2+1\right)^2} \qquad \square\end{aligned}$$

f. *Algebra, Power Rule, and Chain Rule.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2-1}{x^4-1}\right) = \frac{d}{dx}\left(\frac{x^2-1}{\left(x^2-1\right)\left(x^2+1\right)}\right) = \frac{d}{dx}\left(\frac{1}{x^2+1}\right) = \frac{d}{dx}\left(x^2+1\right)^{-1} \\ &= (-1)\left(x^2+1\right)^{-2} \cdot \frac{d}{dx}\left(x^2+1\right) = \frac{-1}{\left(x^2+1\right)^2} \cdot (2x+0) = \frac{-2x}{\left(x^2+1\right)^2} \qquad \blacksquare\end{aligned}$$

2. Evaluate any *five* (5) of the integrals **a**–**f**. *[20 = 5 × 4 each]*

a. $\displaystyle\int \left(ue^{u+1} + 2^{u+2}\right)\, du$ **b.** $\displaystyle\int_0^2 \frac{v^2+39v-82}{v-2}\, dv$ **c.** $\displaystyle\int \frac{\tan(w)}{1-\sin^2(w)}\, dw$

d. $\displaystyle\int_0^{\pi/2} \frac{\sin(2x)}{\sqrt{1+\sin^2(x)}}\, dx$ **e.** $\displaystyle\int y \arctan\left(y^2\right)\, dy$ **f.** $\displaystyle\int_1^2 z^{40} \ln(z)\, dz$

SOLUTIONS. **a.** *A little algebra, integration by parts, and substitutiom.*

$$\begin{aligned}\int \left(ue^{u+1} + 2^{u+2}\right)\, du &= \int \left(eue^u + 4 \cdot 2^u\right)\, du = e\int ue^u\, du + 4\int 2^u\, du \\ &= e\int ue^u\, du + 4\int e^{\ln(2)\cdot u}\, du\end{aligned}$$

The first part we will do using integration by parts, with $u = u$ (Of course! :-) and $v' = e^u$, so $u' = 1$ and $v = e^u$, and the second part we will do using substitution, with $w = \ln(u) \cdot u$, so $dw = \ln(2)\, du$ and $du = \dfrac{1}{\ln(2)}\, dw$. (For the second part, one could also just invoke the appropriate integral formula, but I can't be bothered to remember it ...) Then:

$$\begin{aligned}\int \left(ue^{u+1} + 2^{u+2}\right)\, du &= e\int ue^u\, du + 4\int e^{\ln(2)\cdot u}\, du \\ &= e\left[ue^u - \int e^u\, du\right] + 4\int e^w \cdot \frac{1}{\ln(2)}\, dw \\ &= e\left[ue^u - e^u\right] + \frac{4}{\ln(2)} e^w + C = ue^{u+1} - e^{u+1} + \frac{4e^{\ln(2)\cdot u}}{\ln(2)} + C \\ &= (u-1)e^{u+1} + \frac{4 \cdot 2^u}{\ln(2)} + C = (u-1)e^{u+1} + \frac{2^{u+2}}{\ln(2)} + C \qquad \square\end{aligned}$$

b. *A little algebra and the power rule.*

$$\begin{aligned}\int_0^2 \frac{v^2+39v-82}{v-2}\, dv &= \int_0^2 \frac{(v-2)(v+41)}{v-2}\, dv = \int_0^2 (v+41)\, dv \\ &= \left.\left(\frac{v^2}{2} + 41v\right)\right|_0^2 = \left(\frac{2^2}{2} + 41 \cdot 2\right) - \left(\frac{0^2}{2} + 41 \cdot 0\right) \\ &= (2+82) - 0 = 84 \qquad \square\end{aligned}$$

c. *A little algebra with trig identities, substitution, and the Power Rule.*

$$\int \frac{\tan(w)}{1-\sin^2(w)}\,dw = \int \frac{\frac{\sin(w)}{\cos(w)}}{\cos^2(w)}\,dw = \int \frac{\sin(w)}{\cos^3(w)}\,dw$$

Substitute $t = \cos(w)$, so $dt = -\sin(w)\,dw$ and $\sin(w)\,dw = (-1)\,dt$.

$$= \int \frac{1}{t^3}(-1)\,dt = -\int t^{-3}\,dt = -\frac{t^{-2}}{-2} + C = \frac{1}{2t^2} + C$$

$$= \frac{1}{2\cos^2(w)} + C = \frac{1}{2}\sec^2(w) + C \qquad \square$$

d. *A little algebra with trig identities, substitution, and the Power Rule.*

$$\int_0^{\pi/2} \frac{\sin(2x)}{\sqrt{1+\sin^2(x)}}\,dx = \int_0^{\pi/2} \frac{2\sin(x)\cos(x)}{\sqrt{1+\sin^2(x)}}\,dx$$

Substitute $u = 1+\sin^2(x)$, so $du = 2\sin(x)\cos(x)\,dx$ and $\begin{array}{ccc} x & 0 & \pi/2 \\ u & 1 & 2 \end{array}$.

$$= \int_1^2 \frac{1}{\sqrt{u}}\,du = \int_1^2 u^{-1/2}\,du = \left.\frac{u^{1/2}}{1/2}\right|_1^2 = 2\sqrt{u}\Big|_1^2$$

$$= 2\sqrt{2} - 2\sqrt{1} = 2\sqrt{2} - 2 \approx 0.8284 \qquad \square$$

e. *Substitution and integration by parts.*

$$\int y\arctan\left(y^2\right)\,dy = \int \arctan(w)\,\frac{1}{2}\,dw$$

Using the substitution $w = y^2$, so $dw = 2y\,dy$ and $y\,dy = \frac{1}{2}\,dw$.

$$= \int \frac{1}{2}\arctan(w)\,dw$$

Now use parts, with $u = \arctan(w)$ and $v' = \frac{1}{2}$, so $u' = \frac{1}{1+w^2}$ and $v = \frac{1}{2}w$.

$$= \frac{1}{2}w\arctan(w) - \int \frac{1}{1+w^2}\cdot\frac{1}{2}w\,dw$$

Now let $u = 1+w^2$, so $du = 2w\,dw$ and $w\,dw = \frac{1}{2}\,du$.

$$= \frac{1}{2}w\arctan(w) - \int \frac{1}{u}\cdot\frac{1}{2}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}w\arctan(w) - \frac{1}{4}\ln(u) + C$$

$$= \frac{1}{2}w\arctan(w) - \frac{1}{4}\ln\left(1+w^2\right) + C$$

$$= \frac{1}{2}y^2\arctan\left(y^2\right) - \frac{1}{4}\ln\left(1+y^4\right) + C \qquad \square$$

f. *Integration by parts and the Power Rule.*

We will use the parts $u = \ln(z)$ and $v' = z^{40}$, so $u' = \frac{1}{z}$ and $v = \frac{z^{41}}{41}$.

$$\begin{aligned}
\int_1^2 z^{40} \ln(z)\, dz &= \left.\frac{z^{41}\ln(z)}{41}\right|_1^2 - \int_1^2 \frac{1}{z} \cdot \frac{z^{41}}{41}\, dz = \left[\frac{2^{41}\ln(2)}{41} - \frac{1^{41}\ln(1)}{41}\right] - \frac{1}{41}\int_1^2 z^{40}\, dz \\
&= \left[\frac{2^{41}\ln(2)}{41} - \frac{1 \cdot 0}{41}\right] - \frac{1}{41} \cdot \left.\frac{z^{41}}{41}\right|_1^2 = \frac{2^{41}\ln(2)}{41} - \left[\frac{2^{41}}{41^2} - \frac{1^{41}}{41^2}\right] \\
&= \frac{2^{41}\ln(2)}{41} - \frac{2^{41}-1}{41^2} \approx \texttt{[Calculator screams.]} \qquad \blacksquare
\end{aligned}$$

3. Do any *five* (5) of **a**–**h**. *[20 = 5 × 4 each]*

a. Use the limit definition of the derivative to show that $\frac{d}{dx}x^4 = 4x^3$.

b. Compute $\lim_{x \to 0} x^{-1/2} \sin\left(x^2\right)$.

c. What is the minimum possible perimeter of a rectangle with area 25 m^2?

d. Sketch the solid obtained by revolving $y = 1 - x^2$, for $-1 \le x \le 1$, about the y-axis, and find the volume of this solid.

e. Find any and all vertical and horizontal asymptotes of $y = \arctan\left(x^2\right)$.

f. Find all the local maximum and minimum values of $y = xe^{-x^2}$ on $(-\infty, \infty)$.

g. Use the ε–δ definition of limits to verify that $\lim_{x \to 0} x\sin(x) = 0$.

h. Sketch the region whose bottom border is given by $y = \frac{x^2}{2}$ for $0 \le x \le 2$, and whose top border is given by $y = 2x$ for $0 \le x \le 1$ and by $y = 2$ for $1 \le x \le 2$, and find the area of this region.

SOLUTIONS. **a.** We plug $f(x) = x^4$ into the limit definition of the derivative and go:

$$\begin{aligned}
\frac{d}{dx}x^4 &= \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\
&= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h \to 0} \left(4x^3 + 6x^2h + 4xh^2 + h^3\right) \\
&= 4x^3 + 6x^2 \cdot 0 + 4x \cdot 0^2 + 0^3 = 4x^3 \qquad \square
\end{aligned}$$

b. *For those who noticed:* $\lim_{x \to 0} x^{-1/2} \sin\left(x^2\right)$ is a regular, *i.e.* two-sided, limit, but $x^{-1/2} \sin\left(x^2\right) = \frac{\sin\left(x^2\right)}{\sqrt{x}}$ is undefined for $x < 0$. Thus the given limit is undefined too. $\square$

b. *For those who didn't notice, but just went on to compute the limit.* What follows is the computation of the one-sided limit $\lim_{x\to 0^+} x^{-1/2}\sin\left(x^2\right)$, but no points were taken off for proceeding as if the two-sided limit made sense. We will use l'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to 0^+} x^{-1/2}\sin\left(x^2\right) &= \lim_{x\to 0^+} \frac{\sin\left(x^2\right) \to 0}{x^{1/2} \quad \to 0} = \lim_{x\to 0^+} \frac{\frac{d}{dx}\sin\left(x^2\right)}{\frac{d}{dx}x^{1/2}} = \lim_{x\to 0^+} \frac{\cos\left(x^2\right)\cdot \frac{d}{dx}x^2}{\frac{1}{2}x^{-1/2}} \\
&= \lim_{x\to 0^+} \frac{2x\cos\left(x^2\right)}{\frac{1}{2x^{1/2}}} = \lim_{x\to 0^+} 2x\cos\left(x^2\right)\cdot 2x^{1/2} = \lim_{x\to 0^+} 4x^{3/2}\cos\left(x^2\right) \\
&= 4\cdot 0^{3/2}\cdot\cos\left(0^2\right) = 4\cdot 0\cdot 1 = 0 \qquad \Box
\end{aligned}$$

c. A rectangle of width w and height h has area $A = wh$ and perimeter $P = 2w + 2h$. Since $A = wh = 25$ in this case, $h = \dfrac{25}{w}$, and so $P = 2w + \dfrac{50}{w}$. Note that $0 < w < \infty$ since $h = \dfrac{25}{w}$ and h must be positive and there are no other restrictions on h and w. Then

$$\begin{aligned}
\lim_{w\to 0} P &= \lim_{w\to 0}\left(2w + \frac{50}{w}\right) = 0 + \infty = \infty \\
\text{and } \lim_{w\to\infty} P &= \lim_{w\to\infty}\left(2w + \frac{50}{w}\right) = \infty + 0 = \infty\,,
\end{aligned}$$

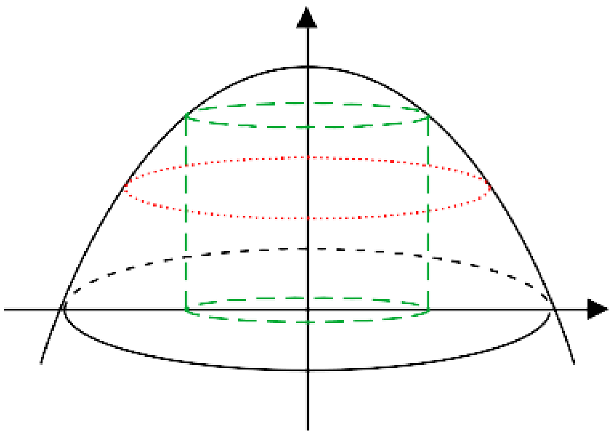
so the minimum length of the perimeter P does not occur at the extremes of the interval. Since $P = 2w + \dfrac{50}{w}$ is defined and differentiable for $0 < w < \infty$, the minimum must occur at a critical point inside the interval.

$$\frac{dP}{dw} = \frac{d}{dw}\left(2w + \frac{50}{w}\right) = 2 - \frac{50}{w^2} = 0 \iff 2w^2 = 50 \iff w^2 = 25 \iff w = \pm 5$$

$w = -5$ is not a possible width, so the only critical point, and hence minimum, is $w = 5$. We can make an additional check that this is a minimum by observing that $\dfrac{dP}{dw} = 2 - \dfrac{50}{w^2}$ is < 0 (so P is decreasing) when $0 < w < 5$ and is > 0 (so P is increasing) when $w > 5$.

It follows that the rectangle with minimum perimeter that has area $25\ m^2$ has width $w = 5\ m$ and height $h = \dfrac{25}{5} = 5\ m$ – it's a square! – and that minimum perimeter is therefore $P = 2\cdot 5 + 2\cdot 5 = 10 + 10 = 20\ m$. $\Box$

d. Note that to get a solid, rather than just a surface, we need to define a region. In this case, the intent was to have $y = 0$, *i.e.* the x-axis, be the bottom of the region. Here is a sketch of the solid:



To help with the solutions below, the sketch includes a disk cross-section outlined in dots and a cylindrical shell cross-section outlined in long dashes. We will find the volume of the solid using both the disk/washer method or the cylindrical shell method. (One suffices for an answer on the exam, of course. :-)

i. Disk/washer method. The variable perpendicular to the disks here is y, so we will use it as the basic variable. Note that our original curve, $y = 1-x^2$ for $-1 \leq x \leq 1$, has $0 \leq y \leq 1$. The disk at y has radius $r = x = \sqrt{1-y}$ and hence area $\pi r^2 = \pi \left(\sqrt{1-y}\right)^2 = \pi(1-y)$. It follows that the volume of the solid is:

$$\begin{aligned} V &= \int_0^1 \pi r^2 \, dy = \pi \int_0^1 (1-y) \, dy = \pi \left(y - \frac{y^2}{2} \right) \Big|_0^1 \\ &= \pi \left(1 - \frac{1^2}{2} \right) - \pi \left(0 - \frac{0^2}{2} \right) = \pi \cdot \frac{1}{2} - \pi \cdot 0 = \frac{\pi}{2} \end{aligned}$$

ii. Cylindrical shell method. The variable perpendicular to the shells here is x, so we will use it as the basic variable. Note that our original curve, $y = 1 - x^2$ for $-1 \leq x \leq 1$, is symmetric about the y-axis, so the region sweeps out the volume twice over when rotated about the y-axis. We can fix this by using only half the region, say the part for $0 \leq x \leq 1$. [Why was this not a problem in the disk method above?] The shell at x, for $0 \leq x \leq 1$ has radius $r = x$ and height $h = y - 0 = 1 - x^2$, and hence area $2\pi r h = \pi x \left(1 - x^2\right) = \pi \left(x - x^3\right)$. It follows that the volume fo the solid is:

$$\begin{aligned} V &= \int_0^1 2\pi r h \, dx = 2\pi \int_0^1 \left(x - x^3\right) \, dx = 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 \\ &= 2\pi \left(\frac{1^2}{2} - \frac{1^4}{4} \right) - 2\pi \left(\frac{0^2}{2} - \frac{0^4}{4} \right) = 2\pi \cdot \frac{1}{4} - 2\pi \cdot 0 = \frac{\pi}{2} \qquad \square \end{aligned}$$

e. $y = \arctan\left(x^2\right)$ is a composition of functions that are defined and continuous everywhere, so it is also defined and continuous everywhere, and therefore has no vertical asymptotes.

As for horizontal asymptotes, recall that $\lim_{t\to+\infty} \arctan(t) = \frac{\pi}{2}$. It follows that

$$\begin{aligned} &\lim_{x\to-\infty} \arctan\left(x^2\right) = \frac{\pi}{2} \quad \text{since } x^2 \to +\infty \text{ as } x \to -\infty \\ \text{and } &\lim_{x\to+\infty} \arctan\left(x^2\right) = \frac{\pi}{2} \quad \text{since } x^2 \to +\infty \text{ as } x \to +\infty. \end{aligned}$$

Thus $y = \arctan\left(x^2\right)$ has a horizontal asymptote of $y = \frac{\pi}{2}$ in both directions. □

f. Observe that $y = xe^{-x^2}$ is defined and continuous, and also differentiable, on $(-\infty, \infty)$, so we don't have to worry about discontinuities, such as vertical asymptotes, complicating things and just focus on critical points.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(xe^{-x^2}\right) = \left[\frac{d}{dx}x\right]e^{-x^2} + x\left[\frac{d}{dx}e^{-x^2}\right] = 1e^{-x^2} + xe^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right) \\ &= e^{-x^2}\left(1 + x(-2x)\right) = e^{-x^2}\left(1 - 2x^2\right) \end{aligned}$$

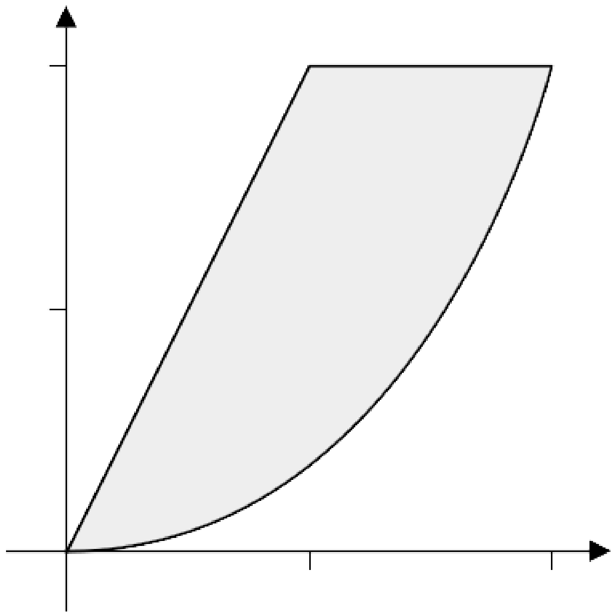
Since $e^{-x^2} > 0$ for all x, $\frac{dy}{dx} = 0 \iff 1 - 2x^2 = 0 \iff x = \frac{\pm 1}{\sqrt{2}}$. Moreover, $\frac{dy}{dx} > 0$ exactly when $1 - 2x^2 > 0$, which is the case when $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, and $\frac{dy}{dx} < 0$ exactly when $1 - 2x^2 < 0$, which is the case when $x < \frac{-1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$.

It follows that the graph of $y = xe^{-x^2}$ is decreasing when $x < \frac{-1}{\sqrt{2}}$, increasing when $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, and decreasing again when $x > \frac{1}{\sqrt{2}}$. Thus $x = \frac{-1}{\sqrt{2}}$ is a local minimum and $x = \frac{1}{\sqrt{2}}$ is a local maximum. The values of $y = xe^{-x^2}$ at these points are $y = \frac{-e^{-1/2}}{\sqrt{2}} = \frac{-1}{\sqrt{2e}}$ and $y = \frac{e^{-1/2}}{\sqrt{2}} = \frac{1}{\sqrt{2e}}$, respectively. □

g. We will use the ε–δ definition of limits to verify that $\lim_{x\to 0} x\sin(x) = 0$. Observe that because $|\sin(x)| \le 1$ for all x, we have $|x\sin(x) - 0| = |x\sin(x)| \le |x| = |x - 0|$.

Suppose now that we are given an $\varepsilon > 0$. Let $\delta = \varepsilon$, so $\delta > 0$ and if $|x - 0| < \delta$, then $|x\sin(x) - 0| \le |x| = |x - 0| < \delta = \varepsilon$, *i.e.* $|x\sin(x) - 0| < \varepsilon$. By the ε–δ definition of limits, it follows that $\lim_{x\to 0} x\sin(x) = 0$. □

h. Here is a sketch of the region in question:



We compute the area of this region as usual, $A = \int_0^2 (\text{upper} - \text{lower})\, dx$, with the small caveat that we have to break the integral up according to the change in what the upper border is:

$$
\begin{aligned}
A &= \int_0^2 (\text{upper} - \text{lower})\, dx = \int_0^1 \left(2x - \frac{x^2}{2}\right) dx + \int_1^2 \left(2 - \frac{x^2}{2}\right) dx \\
&= \left(2 \cdot \frac{x^2}{2} - \frac{1}{2} \cdot \frac{x^3}{3}\right)\Big|_0^1 + \left(2x - \frac{1}{2} \cdot \frac{x^3}{3}\right)\Big|_1^2 = \left(x^2 - \frac{x^3}{6}\right)\Big|_0^1 + \left(2x - \frac{x^3}{6}\right)\Big|_1^2 \\
&= \left(1^2 - \frac{1^3}{6}\right) - \left(0^2 - \frac{0^3}{6}\right) + \left(2 \cdot 2 - \frac{2^3}{6}\right) - \left(2 \cdot 1 - \frac{1^3}{6}\right) \\
&= \left(1 - \frac{1}{6}\right) - 0 + \left(4 - \frac{4}{3}\right) - \left(2 - \frac{1}{6}\right) = \frac{5}{6} + \frac{8}{3} - \frac{11}{6} = \frac{16}{6} - \frac{6}{6} = \frac{10}{6} = \frac{5}{3} \qquad \blacksquare
\end{aligned}
$$

4. Find the domain as well as any (and all) intercepts, vertical and horizontal asymptotes, intervals of increase, decrease and concavity, and maximum, minimum, and inflection points of $h(x) = \dfrac{x}{1-x^2}$, and sketch its graph based on this information. *[12]*

SOLUTION. *i. Domain.* $h(x) = \dfrac{x}{1-x^2}$ is defined for all x except where $1-x^2=0$, *i.e.* when $x = \pm 1$. The domain of $h(x)$ is therefore $\{\, x \in \mathbb{R} \mid x \neq \pm 1 \,\} = (-\infty,-1) \cup (-1,1) \cup (1,\infty)$.

ii. Intercepts. Since $h(0) = \dfrac{0}{1-0^2} = 0$, $h(x)$ has a y-intercept of $y=0$.

As $h(x) = \dfrac{x}{1-x^2} = 0 \iff x = 0$, $h(x)$ has an x-intercept of $x=0$. Note that this is also the y-intercept.

iii. Vertical asymptotes. $h(x)$ is continuous and differentiable wherever it is defined, since it is a composition of continuous and differentiable functions, so the only places there might be vertical asymptotes would be at $x = \pm 1$, where $h(x)$ is undefined. We take limits from each side at both of these points to check for vertical asymptotes:

$$\lim_{x\to -1^-} h(x) = \lim_{x\to -1^-} \frac{x}{1-x^2} \begin{matrix} \to -1^- \\ \to 0^- \end{matrix} = +\infty$$

$$\lim_{x\to -1^+} h(x) = \lim_{x\to -1^+} \frac{x}{1-x^2} \begin{matrix} \to -1^+ \\ \to 0^+ \end{matrix} = -\infty$$

$$\lim_{x\to +1^-} h(x) = \lim_{x\to +1^-} \frac{x}{1-x^2} \begin{matrix} \to +1^- \\ \to 0^+ \end{matrix} = +\infty$$

$$\lim_{x\to +1^+} h(x) = \lim_{x\to +1^+} \frac{x}{1-x^2} \begin{matrix} \to +1^- \\ \to 0^- \end{matrix} = -\infty$$

It follows that $h(x)$ has vertical asymptotes at both $x=-1$ and $x=+1$. At both points $h(x)$ approaches $+\infty$ from the left and approaches $-\infty$ from the right.

iv. Horizontal asymptotes. We take limits as $x \to \pm\infty$ to check for horizontal asymptotes, with a little help from l'Hôpital's Rule:

$$\lim_{x\to -\infty} h(x) = \lim_{x\to -\infty} \frac{x}{1-x^2} \begin{matrix} \to -\infty \\ \to -\infty \end{matrix} = \lim_{x\to -\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}\left(1-x^2\right)} = \lim_{x\to -\infty} \frac{1}{-2x} \begin{matrix} \to 1 \\ \to +\infty \end{matrix} = 0^+$$

$$\lim_{x\to +\infty} h(x) = \lim_{x\to +\infty} \frac{x}{1-x^2} \begin{matrix} \to +\infty \\ \to -\infty \end{matrix} = \lim_{x\to +\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}\left(1-x^2\right)} = \lim_{x\to +\infty} \frac{1}{-2x} \begin{matrix} \to 1 \\ \to -\infty \end{matrix} = 0^-$$

Thus $h(x)$ has $y=0$ as a horizontal asymptote in both directions, which it approaches from above on the left and from below on the right.

v. Intervals of increase and decrease and maximum and minimum points. As usual, we take the derivative and see what it does:

$$h'(x) = \frac{d}{dx}\left(\frac{x}{1-x^2}\right) = \frac{\left[\frac{d}{dx}x\right]\left(1-x^2\right) - x\left[\frac{d}{dx}\left(1-x^2\right)\right]}{\left(1-x^2\right)^2} = \frac{1\left(1-x^2\right) - x(-2x)}{\left(1-x^2\right)^2}$$
$$= \frac{1-x^2+2x^2}{\left(1-x^2\right)^2} = \frac{1+x^2}{\left(1-x^2\right)^2}$$

$h'(x)$ fails to be defined exactly where $h(x)$ fails to be defined, namely at $x = \pm 1$. Note that since both the numerator and denominator of $h'(x) = \dfrac{1+x^2}{\left(1-x^2\right)^2}$ are positive for all x where $h(x)$ is defined, $h'(x) > 0$ for all $x \neq \pm 1$, and so $h(x)$ is increasing for all $x \neq \pm 1$. Thus $h(x)$ has no critical points and hence no maxima or minima. As usual, we summarize this information in a table:

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$h'(x)$	$+$	undef	$+$	undef	$+$
$h(x)$	$\uparrow$	undef	$\uparrow$	undef	$\uparrow$

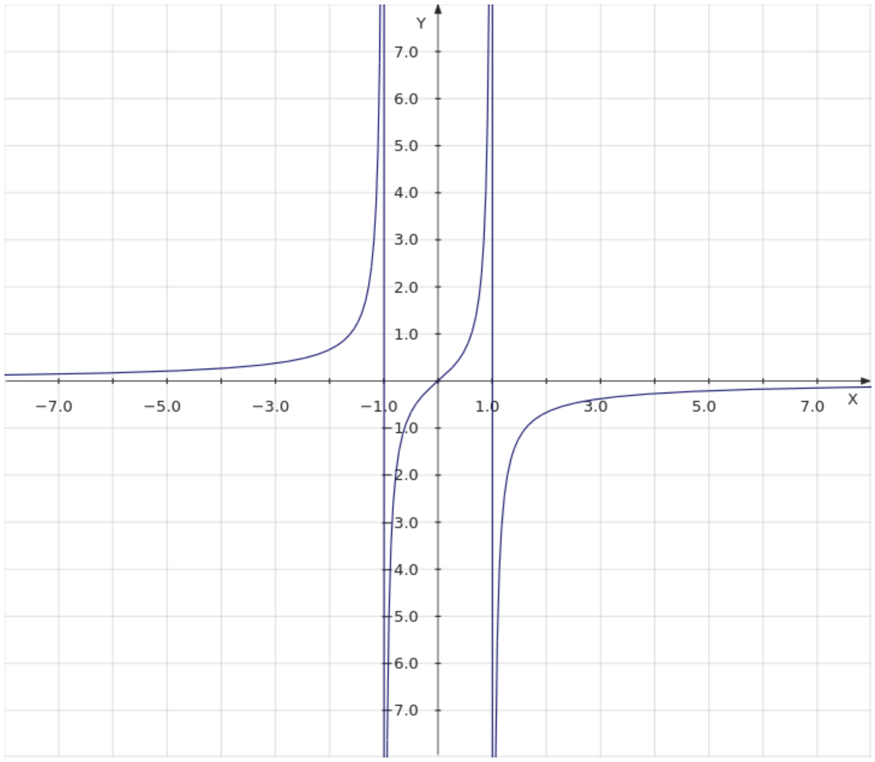
vi. Intervals of concavity and inflection points. As usual, we compute the second derivative and take it from there:

$$\begin{aligned}
h''(x) &= \frac{d}{dx}\left(\frac{1+x^2}{\left(1-x^2\right)^2}\right) = \frac{\left[\frac{d}{dx}\left(1+x^2\right)\right]\left(1-x^2\right)^2 - \left(1+x^2\right)\left[\frac{d}{dx}\left(1-x^2\right)^2\right]}{\left(\left(1-x^2\right)^2\right)^2} \\
&= \frac{2x\left(1-x^2\right)^2 - \left(1+x^2\right)\cdot 2\left(1-x^2\right)\cdot\frac{d}{dx}\left(1-x^2\right)}{\left(1-x^2\right)^4} \\
&= \frac{2x\left(1-x^2\right)^2 - 2\left(1+x^2\right)\left(1-x^2\right)(-2x)}{\left(1-x^2\right)^4} = \frac{2x\left(1-x^2\right)+4x\left(1+x^2\right)}{\left(1-x^2\right)^3} \\
&= \frac{2x-2x^3+4x+4x^3}{\left(1-x^2\right)^3} = \frac{6x+2x^3}{\left(1-x^2\right)^3} = \frac{2x\left(3+x^2\right)}{\left(1-x^2\right)^3}
\end{aligned}$$

Observe that $h''(x)$ is undefined exactly where $h(x)$ and $h'(x)$ are undefined, namely at $x = \pm 1$. As $3 + x^2 > 0$ for all x, $h''(x) = 0$ exactly when $x = 0$. Since $2x$ is positive or negative exactly as x is positive or negative, and $\left(1-x^2\right)^3$ is positive or negative exactly when $1 - x^2$ is positive or negative, *i.e.* when $-1 < x < 1$ and when $|x| > 1$, respectively, we have that $h''(x) = \dfrac{2x\left(3+x^2\right)}{\left(1-x^2\right)^3}$ is positive when $x < -1$, negative when $-1 < x < 0$, positive when $0 < x < 1$, and negative when $x > 1$. This means that the original function $h(x)$ is concave up when $x < -1$, concave down when $-1 < x < 0$, has an inflection point at $x = 0$, is concave up when $0 < x < 1$, and is concave down when $x > 1$. As usual, we summarize this information in a table:

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\infty)$
$h''(x)$	$+$	undef	$-$	0	$+$	undef	$-$
$h(x)$	$\smile$	undef	$\frown$	infl. pt.	$\smile$	undef	$\frown$

vii. Graph. It's a cheat, but here is the graph of $h(x) = \dfrac{x}{1-x^2}$, as drawn by a program called `kmplot`:

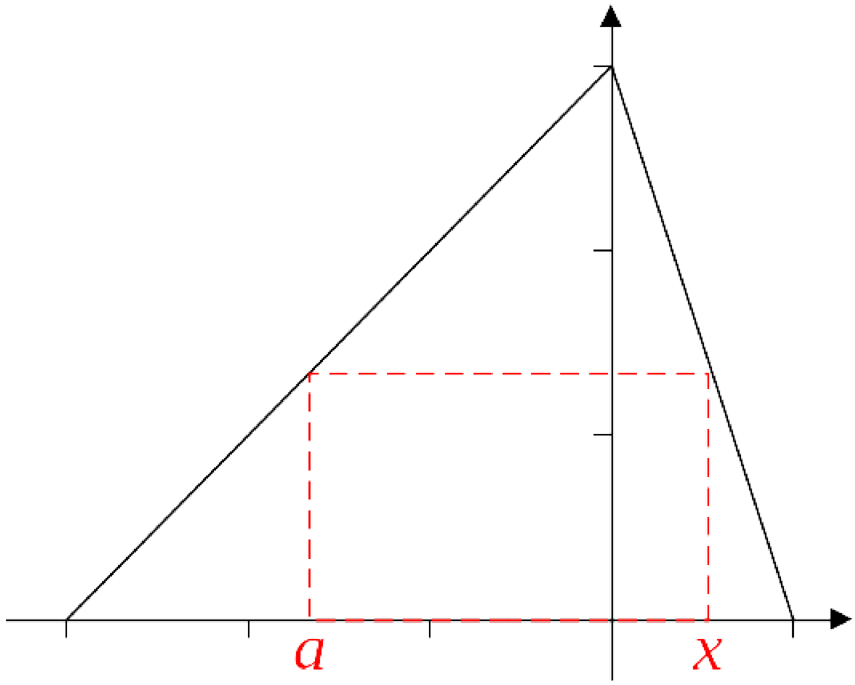


■

Part Yeti. Do any *two* (2) of **5–7**. *[Subtotal = 28 = 2 × 14 each]*

5. A rectangle has its base on the x-axis and its top side runs from the line $y = x + 3$ on the left to the line $y = 3 - 3x$ on the right. Find the maximum area of such a rectangle.

SOLUTION. Here is a sketch of the setup, with the right edge of the rectangle at x for some $0 \le x \le 1$ and the left edge at a for some $-3 \le a \le 0$:



The rectangle with it's right edge at x for some $0 \le x \le 1$ has its top edge at $y = 3-3x$. If the left edge is at a for some $-3 \le a \le 0$, then we must have $a + 3 = y = 3 - 3x$, so $a = -3x$. It follows that the rectangle has width $w = x - (-3x) = x + 3x = 4x$ and height $h = y - 0 = 3 - 3x$, and hence has area $A(x) = wh = 4x(3 - 3x) = 12x - 12x^2$. Our task is to maximize $A(x)$ for $0 \le x \le 1$. Note that $A(x)$ is defined and differentiable for all x, so we only need to check the values of $A(x)$ at the endpoints of the interval $[0, 1]$ and at any critical points in the interval.

At the endpoints of the interval $[0, 1]$ we have $A(0) = 12 \cdot 0 - 12 \cdot 0^2 = 0 - 0 = 0$ and $A(1) = 12 \cdot 1 - 12 \cdot 1^2 = 12 - 12 = 0$, respectively. It remains to check any critical points in the interval:

$$A'(x) = \frac{d}{dx}\left(12x - 12x^2\right) = 12 - 12 \cdot 2x = 12(1 - 2x) = 0 \iff x = \frac{1}{2}$$

The only critical point, at $x = \frac{1}{2}$, is indeed in the interval $[0, 1]$. Since

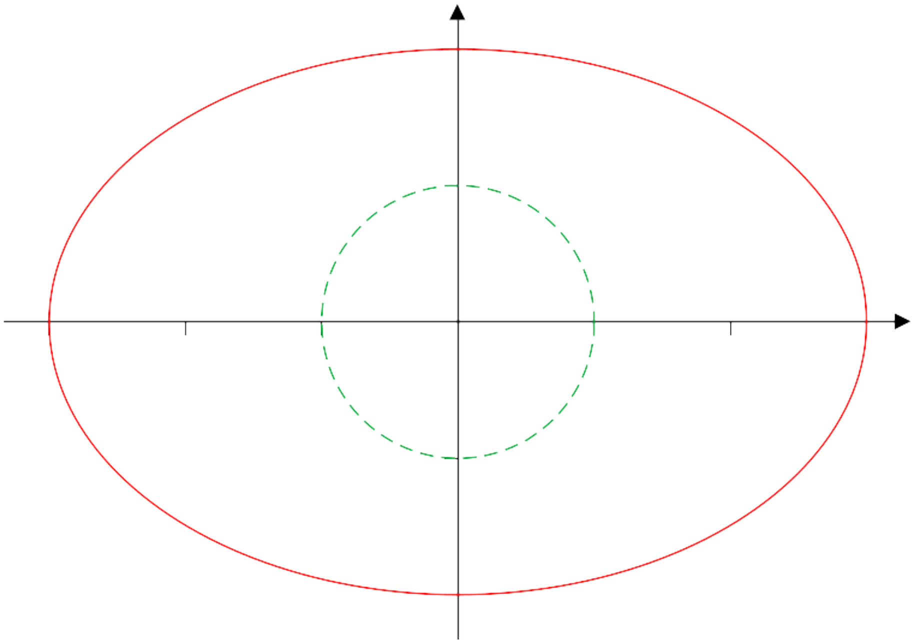
$$A\left(\frac{1}{2}\right) = 12 \cdot \frac{1}{2} - 12\left(\frac{1}{2}\right)^2 = 6 - 12 \cdot \frac{1}{4} = 6 - 3 = 3$$

is greater than the values of $A(x)$ at the endpoints, the greatest possible of area of a rectangle in the given setup is 3. ■

6. Sketch the ellipse given by $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and find its area.

Hint. The integral that would compute the area of an unit circle might be of interest.

Solution. Here is a sketch of the ellipse, with a bonus unit circle $x^2 + y^2 = 1$ inside it:



We will compute the area using two different methods.

i. Using calculus and the hint. We'll set up the integral for computing the area of the ellipse and then try to figure out how to evaluate it. Keeping the hint in mind, note that the integral that computes the area of the unit circle is:

$$U = \int_{-1}^{1} (\text{upper} - \text{lower})\, dx = \int_{-1}^{1} \left(\sqrt{1-x^2} - (-1)\sqrt{1-x^2} \right) dx = 2 \int_{-1}^{1} \sqrt{1-x^2}\, dx$$

Since the area of an unit circle is $\pi 1^2 = \pi$, it follows that $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}$.

Observe that x runs from -3 to 3 for the ellipse, so the area integral should have the form $A = \int_{-3}^{3} (\text{upper} - \text{lower})\, dx$. To find the equations for the upper and lower boundaries, we need to solve the equation giving the ellipse for y:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \implies \frac{y^2}{4} = 1 - \frac{x^2}{9} \implies y^2 = 4\left(1 - \frac{x^2}{9}\right) \implies y = \pm 2\sqrt{1 - \frac{x^2}{9}}$$

It follows that the area of the ellipse is given by:

$$E = \int_{-3}^{3} \left(2\sqrt{1 - \frac{x^2}{9}} - (-2)\sqrt{1 - \frac{x^2}{9}} \right) dx = 4 \int_{-3}^{3} \sqrt{1 - \frac{x^2}{9}}\, dx$$

Following the implications of the hint, we will try to simplify the integral to make it look like the integral that computes the area of the circle. Specifically, we will use the substitution $u = \dfrac{x}{3}$, so $du = \frac{1}{3}\, dx$ and $dx = 3\, du$, and change the limits as we go along: $\begin{matrix} x & -3 & 3 \\ u & -1 & 1 \end{matrix}$. Note also that $\dfrac{x^2}{9} = u^2$. Then

$$E = 4 \int_{-3}^{3} \sqrt{1 - \frac{x^2}{9}}\, dx = 4 \int_{-1}^{1} \sqrt{1-u^2}\, 3\, du = 12 \int_{-1}^{1} \sqrt{1-x^2}\, dx = 12 \cdot \frac{\pi}{2} = 6\pi\,,$$

so the area of the ellipse is 6π.

ii. Using linear algebra and a little inspiration from the hint. Consider the sketch of the unit circle and the ellipse above. If you stretch the unit circle away from the origin, by a factor of 3 horizontally and by a factor of 2 vertically, you should get a curve that looks like the ellipse. In fact, you do get the ellipse:

Suppose the circle has equation $u^2 + v^2 = 1$. [We're using u and v here to avoid confusing ourselves by overusing x and y.] If we stretch it out by a factor of 3 horizontally, we have $x = 3u$, and by a factor of 2 vertically, we have $y = 2v$. For the linear algebra geeks*, this corresponds to the linear transformation given by the matrix equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$. Anyway, we then have $u = \dfrac{x}{3}$ and $v = \dfrac{y}{2}$, so the stretched out curve satisfies

$$\frac{x^2}{9} + \frac{y^2}{4} = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = u^2 + v^2 = 1\,,$$

i.e. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, so it is the ellipse.

How does the stretch affect areas? We can test that by seeing what it does to an unit square with side parallel to the axes: if we stretch it by a factor of 3 horizontally and factor of 2 vertically, we get a rectangle that is 3 units wide and 2 units tall, so it has an area of $3 \cdot 2 = 6$, which is 6 times the area of the unit square.

It follows that the area of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ should be 6 times the area of the unit circle, that is, 6π. ■

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* Should we call them *linearati*, just to make it seem classier?

7. Sand is poured onto a level floor at a constant rate of 100 $L/min = 0.1\ m^3/min$, and at any given instant it forms a conical pile with the height equal to the radius of the base. Compute the rate of change of each of the following at the instant that the height of the cone is 1 m:

i. The height of the cone. *[5]*
ii. The area of the circular base of the cone. *[3]*
iii. The surface area of the rest of the cone. *[6]*

NOTE. The volume of a cone with base radius r and height h is $V = \frac{1}{3}\pi r^2 h$ and its surface area (not counting the base) is $S = \pi r\sqrt{r^2 + h^2}$.

SOLUTION. Since $h = r$ for the conical pile of sand, its volume is $V = \frac{1}{3}\pi r^3$ and its surface area, not counting the base, is $S = \pi r\sqrt{r^2 + r^2} = \pi r\sqrt{2r^2} = \pi r \cdot \sqrt{2}r = \sqrt{2}\pi r^2$. We are given that $\frac{dV}{dt} = 0.1\ m^3/min$ and asked to compute various other rates of change related to the conical sand pile at the instant that the height, and hence the base radius, of the conical pile is 1 m.

i. Since we always have $h = r$ in this setup, $\frac{dh}{dt} = \frac{dr}{dt}$ at any given instant. It follows from

$$0.1 = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi r^3\right) = \frac{\pi}{3}\left(\frac{d}{dr}r^3\right)\frac{dr}{dt} = \frac{\pi}{3} \cdot 3r^2 \cdot \frac{dr}{dt} = \pi r^2 \frac{dr}{dt},$$

that at any given instant $\frac{dr}{dt} = \frac{0.1}{\pi r^2}$. Thus, at the instant that $h = r = 1\ m$, we have $\left.\frac{dh}{dt}\right|_{h=1} = \left.\frac{dr}{dt}\right|_{r=1} = \frac{0.1}{\pi 1^2} = \frac{0.1}{\pi} \approx 0.0318\ m/min$. That is, the height of the conical pile is increasing at a rate of $\frac{0.1}{\pi}\ m/min \approx 3.18\ cm/min$ at the instant that the pile is 1 m high.

ii. The area of a circle of radius r is $A = \pi r^2$, so its rate of change with time is

$$\frac{dA}{dt} = \frac{d}{dt}\pi r^2 = \pi\left(\frac{d}{dr}r^2\right)\frac{dr}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt} = 2\pi r\frac{dr}{dt}.$$

Recall from the solution to part *i* above that $\frac{dr}{dt} = \frac{0.1}{\pi r^2}$. It follows that at the instant that $r = h = 1\ m$, we have

$$\begin{aligned}\left.\frac{dA}{dt}\right|_{h=1} &= \left.\frac{dA}{dt}\right|_{r=1} = \left.2\pi r\frac{dr}{dt}\right|_{r=1} = \left.2\pi r\frac{0.1}{\pi r^2}\right|_{r=1} \\ &= \left.\frac{0.2}{r}\right|_{r=1} = \frac{0.2}{1} = 0.2 \cdot \pi \approx 0.2\ m^2/min.\end{aligned}$$

That is, the area of the circular base of the sand pile is growing at the rate of $0.2\ m^2/min = 2000\ cm^2/min$ at the instant that the pile is 1 m high.

iii. As noted above the surface area of the conical sand pile, not counting the base, is $S = \sqrt{2}\pi r^2$. Thus

$$\frac{dS}{dt} = \frac{d}{dt}\left(\sqrt{2}\pi r^2\right) = \sqrt{2}\pi\left(\frac{d}{dr}r^2\right)\frac{dr}{dt} = \sqrt{2}\pi \cdot 2r \cdot \frac{dr}{dt} = 2\sqrt{2}\pi r\frac{dr}{dt}\,.$$

Recall from the solution to part i above that $\frac{dr}{dt} = \frac{0.1}{\pi r^2}$. It follows that at the instant that $r = h = 1\ m$, we have

$$\begin{aligned}\left.\frac{dS}{dt}\right|_{h=1} &= \left.\frac{dS}{dt}\right|_{r=1} = \left.2\sqrt{2}\pi r\frac{dr}{dt}\right|_{r=1} = \left.2\sqrt{2}\pi r\frac{0.1}{\pi r^2}\right|_{r=1} \\ &= \left.\frac{0.2\cdot\sqrt{2}}{r}\right|_{r=1} = \frac{0.2\cdot\sqrt{2}}{1} = 0.2\cdot\sqrt{2} \approx 0.2828\ m^2/min\,.\end{aligned}$$

That is, the surface area of the conical sand pile, not counting the base, is growing at a rate of $0.2\cdot\sqrt{2}\ m^2/min \approx 0.2828\ m^2/min = 2828\ cm^2/min$ at the instant that the pile is $1\ m$ high. ■

[Total = 100]

Part Zombie. *Bonus problems!* If you feel like it, do one or both of these.

$\sqrt[2]{64}$. If $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$, what does $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$ add up to? *[1]*

ANSWER. It adds up to $\frac{\pi^2}{8}$. You figure out why! :-) ■

$\sqrt[3]{729}$. Write an original poem touching on calculus or mathematics in general. *[1]*

ANSWER. You're on your own here! :-) ■

I HOPE THAT YOU ENJOYED THE COURSE. ENJOY THE SUMMER!