Limits - an alternate ε-δ definition

The usual ε-δ definition of limits has the downside of being fairly intricate, making it harder to understand and use

Standard defn: $\lim_{x \to a} f(x) = L$ means

for every $\varepsilon > 0$

there is a $\delta > 0$

such that for every x

if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

We'll give an equivalent definition that breaks down some of the complexity of the above defn. We will "reduce" the problem to that of finding a winning strategy in a game.

Alternate Definition:

(i) The *limit game* for $f(x)$ at $x=a$ with target L is a three-move game with players A and B that works as follows:

Move 1. A chooses an $\varepsilon > 0$.

Move 2. B chooses a $\delta > 0$.

Move 3. A chooses an x with $a-\delta < x < a+\delta$. (ie $|x-a| < \delta$)

We determine the winner by evaluating $f(x)$:

(1) If $L-\varepsilon < f(x) < L+\varepsilon$ (ie $|f(x)-L| < \varepsilon$), then B wins.

(2) If $f(x) \leq L-\varepsilon$ or $f(x) \geq L+\varepsilon$ (ie $|f(x)-L| \geq \varepsilon$), then A wins.

Note: There must be a winner since the winning conditions are complementary. (No ties!)

(ii) $\lim_{x \to a} f(x) = L$ means that B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

→ A way to win, no matter how A plays.

(Conversely, if A has a winning strategy, then $\lim_{x \to a} f(x)$ does not exist, or at least is not equal to L.)

"$\lim_{x \to a} f(x) \neq L$"

Example: Let's use this definition to check that

$$\lim_{x \to 1} (3x+2) = 5.$$

The limit game for $f(x) = 3x+2$ at $x=1$ with target 5 works as follows:

1. A chooses an $\varepsilon > 0$.
2. B responds by choosing a $\delta > 0$.
3. A responds, in turn, by choosing an x such that $1-\delta < x < 1+\delta$ (ie $|x-1| < \delta$).

[For $\lim_{x \to 1} (3x+2) = 5$, we need to have B have a winning strategy.]

B wins if $5-\varepsilon < 3x+2 < 5+\varepsilon$ (ie $|(3x+2)-5| < \varepsilon$)
(where it's the x that A plays ~~on~~ move 3).

A winning strategy for B must select a $\delta > 0$ in response to $\varepsilon > 0$ such that no matter how player A chooses an x with $1-\delta < x < 1+\delta$, we get $5-\varepsilon < 3x+2 < 5+\varepsilon$.

As with the standard definition, we'll reverse-~~eng~~ineer the necessary $\delta > 0$ from the desired winning condition:

$$5-\varepsilon < 3x+2 < 5+\varepsilon \qquad \text{[the winning condition for } B\text{]}$$

$$\iff -\varepsilon < \underbrace{3x+2-5}_{=\,3x-3} < \varepsilon$$

$$\iff -\varepsilon < 3(x-1) < \varepsilon$$

$$\iff -\frac{\varepsilon}{3} < x-1 < \frac{\varepsilon}{3}$$

$$\iff 1-\frac{\varepsilon}{3} < x < 1+\frac{\varepsilon}{3}$$

So if B can guarantee that $1-\frac{\varepsilon}{3} < x < 1+\frac{\varepsilon}{3}$, then B will win.

<u>Strategy</u>: After A plays $\varepsilon > 0$ on move 1, B responds with $\delta = \frac{\varepsilon}{3}$ on move 2.

Then no matter what x A picks with $\underbrace{1-\delta}_{=\,1-\frac{\varepsilon}{3}} < x < \underbrace{1+\delta}_{=\,1+\frac{\varepsilon}{3}}$, we have B winning.

$\therefore \lim\limits_{x \to 1} (3x+2) = 5$ because B has strategy that always wins.