Taylor Series (and other dirty tricks)

(Section 11.10 in the text)

The big problem one has when trying to take advantage of power series (along with convergence), is how to write familiar functions as power series. This boils down to trying to figure out what each coefficient of x^n needs to be to make the power series add up to whatever function $f(x)$ we want to express as a power series.

$$\text{Suppose } f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots.$$

- Then $f(0) = a_0 + a_1\cdot 0 + a_2\cdot 0^2 + \cdots = a_0$. Thus $a_0 = f(0)$.
- Now, observe that $f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n x^n)$

$$= \sum_{n=0}^{\infty} n a_n x^{n-1} = \cancel{0\, a_0} + a_1 + 2a_2 x + \cdots,$$

so $f'(0) = a_1$. Thus $a_1 = f'(0)$.

- Similarly, $f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\sum_{n=\cancel{0}}^{\infty} n a_n x^{n-1}\right) = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2}$

$$= \cancel{1\cdot(1-1)a_1} + 2\cdot(2-1)a_2 + 3\cdot(3-1)a_3 x + \cdots$$

$$= 2a_2 + 3\cdot 2\cdot a_3 x + \cdots,$$

so $f''(0) = 2a_2$. Thus $a_2 = \frac{f''(0)}{2}$.

- Once more! $f'''(x) = \frac{d}{dx} f''(x) = \frac{d}{dx}\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right)$

$$= \sum_{n=2}^{\infty} \frac{d}{dx}\left(n(n-1) a_n x^{n-2}\right)$$

$$= \sum_{n=2}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

$$= \cancel{2\cdot 1\cdot(2-2)}a_2 + 3\cdot 2\cdot 1\cdot a_3 + 4\cdot 3\cdot 2 a_4 x + \cdots,$$

so $f'''(0) = 3\cdot 2\cdot 1\cdot a_3 = 6a_3$. Thus $a_3 = \frac{f'''(0)}{6}$.

If we denote the k^{th} derivative of $f(x)$ by $f^{(k)}(x)$,

and continue this process, we get the formula

$$f^{(k)}(0) = k!\,a_k \;, \quad \text{so} \quad a_k = \frac{f^{(k)}(0)}{k!}.$$

This gives us <u>Taylor's formula</u>:

the <u>Taylor series</u> of a function $f(x)$ (at $a=0$) is the power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

[Here we follow the conventions that $0!$ and that the 0^{th} derivative of $f(x)$ is the function itself. These conventions exist in large part to make this formula nicer, by not having to write exceptions for $n=0$.]

With <u>rare</u> exceptions, when the Taylor series of a function converges, it converges to $f(x)$, ie $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(x)$.

[The first discovered* exception is the function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

* Damn that guy Cauchy...

This turns out to be differentiable at 0 (as many times as you like), and $f^{(n)}(0) = 0$ for all $n \geq 0$. Thus the Taylor series converges for all x and is always equal to 0, unlike the function, which $=0$ only at $x=0$.]

A nice example: $f(x) = e^x$

Since $\frac{d}{dx} e^x = e^x$, it's not hard to see that $f^{(n)}(x) = e^x$ for all $n \geq 0$. It follows that $f^{(n)}(0) = e^0 = 1$ for all $n \geq 0$, and so the Taylor series (at 0) of e^x is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This series converges to e^x for all x.

One small use of this series is to approximate the number e. $e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}$, so if you want to approximate e, all you have to do is take a big enough partial sum $\sum_{n=0}^{k} \frac{1}{n!}$.

Another nice example: $f(x) = \sin(x)$

Then $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f^{(4)}(x) = -(-\sin(x)) = \sin(x)$, and things continue repeating after that. Thus we have

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0
5	$\cos(x)$	1
6	$-\sin(x)$	0
7	$-\cos(x)$	-1
	$\vdots$	

The general pattern is that $f^{(n)}(0) = 0$ if $n = 2k$, and $f^{(n)}(0) = (-1)^k$ if $n = 2k+1$.

Thus the Taylor series (at 0) of $f(x) = \sin(x)$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

This brings us to

<u>Dirty Trick #1:</u> If you know the Taylor series of $f(x)$, and you take its derivative, you get the Taylor series of $f'(x)$.

For example, once we know the Taylor series of $\sin(x)$, we can get the Taylor series for $\cos(x)$ without having to use Taylor's formula:

$$\cos(x) = \frac{d}{dx}\sin(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{(2n+1)} x^{2n}}{\cancel{(2n+1)} \cdot (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

More examples and dirty tricks next time!