Power Series (Section 11.8 in the textbook)

Recall that a polynomial in a variable x is an expression of the form $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$, where the a_i are constants. Polynomials are easy to differentiate and integrate, even if other operations on them are hard on occasion. (Like factoring the fool things...) Our objective will be to write arbitrary functions in a form as close to polynomials as we can, so as to gain the advantages they have in doing calculus. To do this, we do have to give up one nice thing about polynomials: the fact that they are finite expressions. Which brings us to:

Def'n: A power series (in the variable x) is an expression of the form

$$\sum_{n=0}^{\infty} a_n x^n ,$$

where each of the a_n is a constant as far as x is concerned.

(That is, the definition of a_n may depend on n, but it may not depend on x.)

The prototype here is the geometric series with common ratio $r = x$:

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \cdots,$$

where a is some constant.

Because polynomials are finite expressions, they make sense (is can be evaluated) no matter what number you plug in for x. Power series are not guaranteed to do this. For example, the one given above converges only if $|x| < 1$ and diverges if $|x| \geq 1$. (Unless $a = 0$, in which case it converges - to 0 - for all x. ☺) When dealing with power series, it will be important to know for which values of x they converge and for which they do not. Our first line tool for the job will be the Ratio Test.

Example 1: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n \cdot (n+1)} = 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{24} + \frac{x^4}{80} + \cdots$

First, we throw the Ratio Test at this:

$$\lim_{n\to\infty} \left| \frac{\dfrac{x^{n+1}}{2^{n+1}(n+1+1)}}{\dfrac{x^n}{2^n(n+1)}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}(n+2)} \cdot \frac{2^n(n+1)}{x^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{2} \cdot \frac{n+1}{n+2} = \frac{|x|}{2} \lim_{n\to\infty} \frac{n+1}{n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{|x|}{2} \cdot \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}}$$

$= \dfrac{|x|}{2} \cdot \dfrac{1+0}{1+0} = \dfrac{|x|}{2}$. It follows by the Ratio

Test that the series converges if $\frac{|x|}{2} < 1$ (absolutely, too) and diverges if $\frac{|x|}{2} > 1$. If $\frac{|x|}{2} = 1$, the test gives us no information, so we have to use other tests to settle this case: $\frac{|x|}{2} = 1 \iff |x| = 2 \iff x = \pm 2$.

In the case of $x = -2$, the series is

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{2^n}}{\cancel{2^n}(n+1)} = \frac{1}{0+1} - \frac{1}{1+1} + \frac{1}{2+1} - \frac{1}{3+1} + \cdots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots,$$

which is to say it is the alternating harmonic series, which converges by the Alternating Series Test.

In the case of $x = +2$, the series is

$$\sum_{n=0}^{\infty} \frac{\cancel{2^n}}{\cancel{2^n}(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \frac{1}{0+1} + \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots,$$

which is to say it is the harmonic series, which diverges by the p-Test (since $p = 1 \leq 1$).

Thus the series $\sum_{n=0}^{\infty} \frac{x^n}{2^n(n+1)}$ converges for x with $-2 \leq x < 2$, ie $x \in [-2, 2)$, and diverges otherwise.

Example 2: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$

Again, we use the Ratio Test first:

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)x^2}{(2n+3)(2n+2)} \right| = x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = x^2 \cdot 0 = 0 < 1$$

It follows by the Ratio Test that this series converges (absolutely) for all values of x, i.e. for $x \in (-\infty, \infty)$.

In general, a power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges absolutely on some interval of the form $(-r, r)$. If $r < \infty$, then the power series may converge (possibly absolutely, possibly conditionally) at neither endpoint, at one endpoint but not the other, or at both endpoints.

Def'n: This r is called the radius of convergence of the power series. The actual interval of all x for which it converges is its interval of convergence.

In Example 1 we had radius of convergence $r = 2$ and interval of convergence $[-2, 2)$; in Example 2 we had radius of convergence $r = \infty$ & interval $(-\infty, \infty)$.

Notice that if x is inside the interval of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$, we can think of the power series as a function of x. As long as x is inside the radius of convergence, i.e. $x \in (-r, r)$, it is safe to differentiate and integrate it term-by-term, just like a polynomial.

One variation on all this that is useful when dealing with functions, like $\ln(x)$, that are not defined at 0, is a <u>power series about $a \in \mathbb{R}$</u>, i.e. a series of the form $\sum_{n=0}^{\infty} a_n (x-a)^n$.

⌈We could hope - and we'd be right to hope - that $\ln(x)$ could be written as a power series about $a=1$, even if it isn't even defined at 0.⌋

Everything works pretty much the same way as for regular power series (i.e. series about $a=0$), except that if $r<\infty$, the interval of convergence has endpoints $a-r$ and $a+r$, instead of $-r$ and r.