Let's try to rearrange the alternating harmonic series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$, to add up to 1.5. We must use all the terms of the series, but we can add them up in any order we'd like. We'll start by trying to get to 1.5 by adding positive terms

1 is less than $1.5 = \frac{3}{2}$.

$1 + \frac{1}{3}$ is still less than 1.5.

$1 + \frac{1}{3} + \frac{1}{5}$ is greater than 1.5. [Oops! Time to use the negatives...]

$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2}$ is less than 1.5

$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7}$ is less than 1.5

$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9}$ is less than 1.5

$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13}$ is greater than 1.5 [—"—]

$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13}\;\frac{1}{4}$ is less than 1.5

$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13}\;\frac{1}{4} + \frac{1}{15}$ is less than 1.5

$\vdots$

Since the alternating harmonic series does <u>not</u> converge absolutely, both $\sum_{k=0}^{\infty} \frac{1}{2k+1} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$ and $\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ diverge by "adding up to infinity".

Thus, each time the process given has a partial sum that gets over 1.5, and then gets knocked back below by using the next negative term, it will get back above 1.5 after adding enough of the (infinitely many!) remaining positive terms.

It's not hard to see that this process *eventually* uses *every* positive and *every* negative term in the alternating harmonic series. Since the positive and the negative terms we're using get closer and closer to 0, the partial sums in this rearrangement eventually get closer & closer to 1.5. [They will always be as close or closer to 1.5 as the absolute value of the last negative term that was used.]

Thus rearranging the terms of the alternating harmonic series to

$$1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}-\frac{1}{4}+\frac{1}{15}+\frac{1}{17}+\cdots$$

gets it to add up to $1.5=\frac{3}{2}$.

Similar rearrangements can be used to get the series to add up to *any* real number you like or to diverge. The same kind of thing can be done with *any* conditionally convergent series, but *not* with absolutely convergent series.

The curious reader might ask what number the alternating harmonic series converges to if we don't rearrange it. We'll use a little algebra and calculus to figure this out.

First, we'll use the formula for the sum of a geometric series in reverse:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots.$$

Second, we integrate both sides with respect to x.

$$\int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + x^4 - x^5 + \cdots)\,dx$$

[substitute $u=1+x$ etc.]

$$\ln(1+x) = \int 1\,dx - \int x\,dx + \int x^2\,dx - \int x^3\,dx + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C$$

Witness the power of fully operational power series!

Third, if we plug in $x=0$ on both sides, we get $\ln(1+0) = \ln(1) = 0$ on the left, and

$$0 - \frac{0^2}{2} + \frac{0^3}{3} - \frac{0^4}{4} + \cdots + C = 0 + C = C$$

on the right. It follows that $C = 0$.

Fourth, if we now plug in $x=1$ on both sides, we get the alternating harmonic series on the right and $\ln(1+1) = \ln(2)$ on the left. Thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln(2).$$