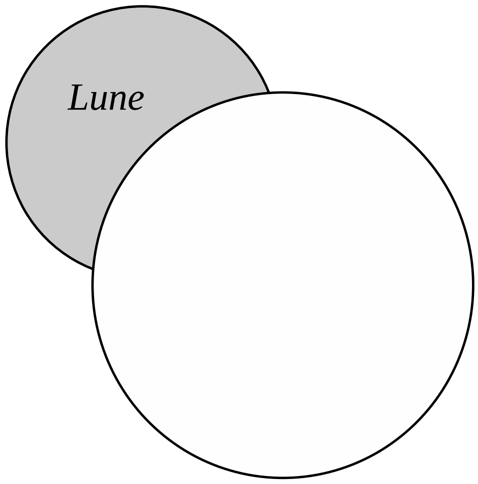
**Mathematics 1120H – Calculus II: Integrals and Series**
Trent University, Winter 2020

**Solutions to Assignment #2**
**Lunes**

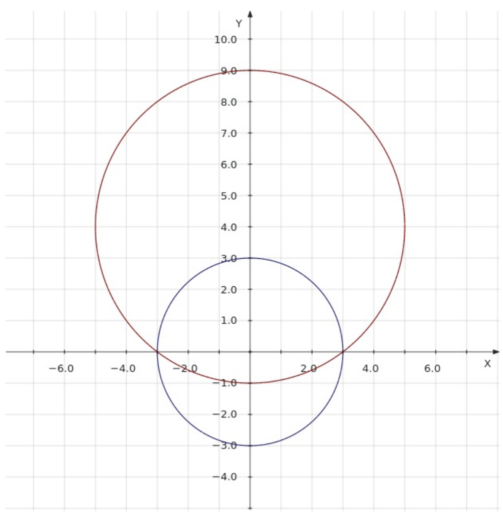
The region inside one but outside the other of two overlapping circles is called a *lune*.




One of the earliest successes in computing areas of non-polygonal plane regions was by Hippocrates of Chios* (*c.* 470–410 B.C.), who found the total area of certain pairs of lunes.

**1.** Find the area of the region that is inside the circle $x^2 + y^2 = 9$ and outside the circle $x^2 + (y - 4)^2 = 25$. *[6]*

Solution. Here is plot of the two circles:




* Not to be confused with his rather better known contemporary, the physician Hippocrates of Cos (*c.* 460–370 B.C.), after whom the Hippocratic Oath is named.

The lune we wish to find the area of is the "smile" below the $x$-axis that is inside the circle $x^2 + y^2 = 9$ and outside the circle $x^2 + (y-4)^2 = 25$. Note that the two circles intersect each other at $-3$ and $3$ on the $x$-axis. Solving for $y$ in each circle (with $y \leq 0$), this boils down to finding the area above $y = -\sqrt{9 - x^2}$ and below $y = 4 - \sqrt{25 - x^2}$ for $-3 \leq x \leq 3$.

$$\begin{aligned}\text{Area} &= \int_{-3}^{3} [\text{upper} - \text{lower}]\, dx = \int_{-3}^{3} \left[\left(4 - \sqrt{25 - x^2}\right) - \left(-\sqrt{9 - x^2}\right)\right]\, dx \\ &= \int_{-3}^{3} 4\, dx - \int_{-3}^{3} \sqrt{25 - x^2}\, dx + \int_{-3}^{3} \sqrt{9 - x^2}\, dx\end{aligned}$$

The first of the three integrals can be handled easily using the Power Rule. For the second integral we will use the trigonometric substitution $x = 5\sin(\theta)$, so $dx = 5\cos(\theta)\, d\theta$, with $\sin(\theta) = \frac{x}{5}$ and $\cos(\theta) = \sqrt{1 - \sin^2(\theta)} = \sqrt{1 - \frac{x^2}{25}}$. Similarly, for the third integral we will use the trigonometric substitution $x = 3\sin(\alpha)$, so $dx = 3\cos(\alpha)\, d\alpha$, with $\sin(\alpha) = \frac{x}{3}$ and $\cos(\alpha) = \sqrt{1 - \sin^2(\alpha)} = \sqrt{1 - \frac{x^2}{9}}$. In both the second and the third, we will keep the old limits and eventually substitute back in the antiderivative before using them.

$$\begin{aligned}\text{Area} &= \int_{-3}^{3} 4\, dx - \int_{-3}^{3} \sqrt{25 - x^2}\, dx + \int_{-3}^{3} \sqrt{9 - x^2}\, dx \\ &= 4x\Big|_{-3}^{3} - \int_{x=-3}^{x=3} \sqrt{25 - 25\sin^2(\theta)}\, 5\cos(\theta)\, d\theta + \int_{x=-3}^{x=3} \sqrt{9 - 9\sin^2(\alpha)}\, 3\cos(\alpha)\, d\alpha \\ &= 4 \cdot 3 - 4 \cdot (-3) - \int_{x=-3}^{x=3} \sqrt{25\cos^2(\theta)}\, 5\cos(\theta)\, d\theta + \int_{x=-3}^{x=3} \sqrt{9\cos^2(\alpha)}\, 3\cos(\alpha)\, d\alpha \\ &= 24 - \int_{x=-3}^{x=3} 5\cos(\theta) \cdot 5\cos(\theta)\, d\theta + \int_{x=-3}^{x=3} 3\cos(\alpha) \cdot 3\cos(\alpha)\, d\alpha \\ &= 24 - 25\int_{x=-3}^{x=3} \cos^2(\theta)\, d\theta + 9\int_{x=-3}^{x=3} \cos^2(\alpha)\, d\alpha\end{aligned}$$

We will now apply the reduction formula for integrating powers of $\cos(t)$, $\int \cos^n(t)\, dt = \frac{1}{n}\cos^{n-1}(t)\sin(t) + \frac{n-1}{n}\int \cos^{n-2}(t)\, dt$.

$$\begin{aligned}\text{Area} &= 24 - 25\int_{x=-3}^{x=3} \cos^2(\theta)\, d\theta + 9\int_{x=-3}^{x=3} \cos^2(\alpha)\, d\alpha \\ &= 24 - 25\left[\frac{1}{2}\cos(\theta)\sin(\theta)\Big|_{x=-3}^{x=3} + \frac{1}{2}\int_{x=-3}^{x=3} 1\, d\theta\right] \\ &\quad + 9\left[\frac{1}{2}\cos(\alpha)\sin(\alpha)\Big|_{x=-3}^{x=3} + \frac{1}{2}\int_{x=-3}^{x=3} 1\, d\alpha\right] \\ &= 24 - \frac{25}{2}\left[\cos(\theta)\sin(\theta) + \theta\right]\Big|_{x=-3}^{x=3} + \frac{9}{2}\left[\cos(\alpha)\sin(\alpha) + \alpha\right]\Big|_{x=-3}^{x=3}\end{aligned}$$

At this point we need to substitute back in terms of $x$. We already noted that $\sin(\theta) = \frac{x}{5}$ and $\cos(\theta) = \sqrt{1 - \frac{x^2}{25}}$ and that $\cos(\alpha) = \sqrt{1 - \sin^2(\alpha)} = \sqrt{1 - \frac{x^2}{9}}$. Note that we also have $\theta = \arcsin\left(\frac{x}{5}\right)$ and $\alpha = \arcsin\left(\frac{x}{3}\right)$. We plug all this in and evaluate:

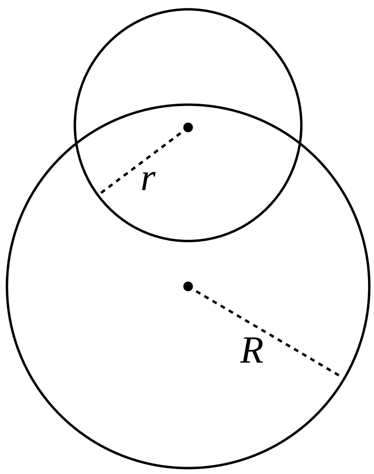
$$\begin{aligned}
\text{Area} &= 24 - \frac{25}{2}\left[\cos(\theta)\sin(\theta) + \theta\right]\Big|_{x=-3}^{x=3} + \frac{9}{2}\left[\cos(\alpha)\sin(\alpha) + \alpha\right]\Big|_{x=-3}^{x=3} \\
&= 24 - \frac{25}{2}\left[\frac{x}{5}\sqrt{1 - \frac{x^2}{25}} + \arcsin\left(\frac{x}{5}\right)\right]\Bigg|_{x=-3}^{x=3} + \frac{9}{2}\left[\frac{x}{3}\sqrt{1 - \frac{x^2}{9}} + \arcsin\left(\frac{x}{3}\right)\right]\Bigg|_{x=-3}^{x=3} \\
&= 24 - \frac{25}{2}\left[\frac{3}{5}\sqrt{1 - \frac{3^2}{25}} + \arcsin\left(\frac{3}{5}\right)\right] + \frac{25}{2}\left[\frac{-3}{5}\sqrt{1 - \frac{(-3)^2}{25}} + \arcsin\left(\frac{-3}{5}\right)\right] \\
&\qquad + \frac{9}{2}\left[\frac{3}{3}\sqrt{1 - \frac{3^2}{9}} + \arcsin\left(\frac{3}{3}\right)\right] - \frac{9}{2}\left[\frac{-3}{3}\sqrt{1 - \frac{(-3)^2}{9}} + \arcsin\left(\frac{-3}{3}\right)\right] \\
&= 24 - \frac{25}{2}\cdot\frac{3}{5}\sqrt{\frac{16}{25}} - \frac{25}{2}\arcsin\left(\frac{3}{5}\right) + \frac{25}{2}\cdot\frac{-3}{5}\sqrt{\frac{16}{25}} + \frac{25}{2}\arcsin\left(\frac{-3}{5}\right) \\
&\qquad + \frac{9}{2}\cdot 1\sqrt{1-1} + \frac{9}{2}\arcsin(1) - \frac{9}{2}\cdot(-1)\sqrt{1-1} - \frac{9}{2}\arcsin(-1) \\
&= 24 - \frac{25}{2}\cdot\frac{3}{5}\cdot\frac{4}{5} - \frac{25}{2}\arcsin\left(\frac{3}{5}\right) - \frac{25}{2}\cdot\frac{3}{5}\cdot\frac{4}{5} - \frac{25}{2}\arcsin\left(\frac{3}{5}\right) \\
&\qquad + \frac{9}{2}\cdot 1\cdot 0 + \frac{9}{2}\cdot\frac{\pi}{2} + \frac{9}{2}\cdot(-1)\cdot 0 - \frac{9}{2}\cdot\left(-\frac{\pi}{2}\right) \\
&= 24 - 12 - 25\arcsin\left(\frac{3}{5}\right) + \frac{9\pi}{2} = 12 + \frac{9\pi}{2} - 25\arcsin\left(\frac{3}{5}\right) \approx 10.0497 \qquad \blacksquare
\end{aligned}$$

**2.** Suppose $R > r > 0$. A circle of radius $r$ has its centre a distance somewhere strictly between $R - r$ and $R$ from the centre of a circle of radius $R$.

**a.** Sketch this arrangement of circles. *[1]*

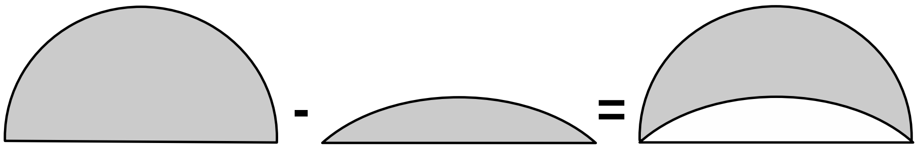
**b.** Find the area of the lune inside the circle of radius $r$ and outside the circle of radius $R$. *[3]*

SOLUTIONS. **a.** Here we are:



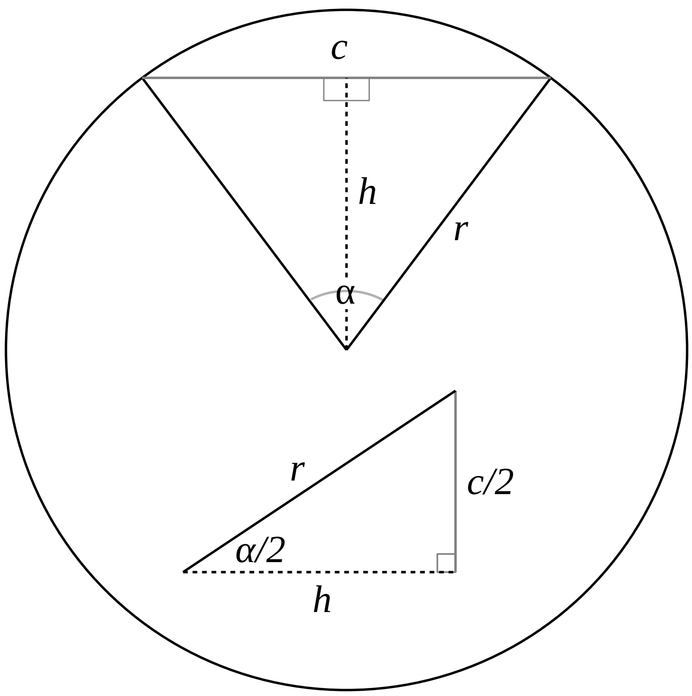

$\square$

**b.** We will forego calculus in favour of geometry, laced with algebra and trigonometry, to find the area of the lune inside the circle of radius $r$ and outside the circle of radius $R$, where $R > r > 0$ and the centre of the circle of radius $r$ is a distance somewhere strictly between $R - r$ and $R$ from the centre of a circle of radius $R$. (The last condition amounts to saying that the centre of the circle of radius $r$ is inside the circle of radius $R$, and that the circle of radius $r$ is not entirely contained inside the circle of radius $R$.) We will try to simplify our task by observing that a lune is a difference of two caps of different circles sharing the same chord:




Our strategy will be to express the area of a cap cut off by a chord of length $c$ from a circle of radius $r$ in terms of $c$ and $r$, and then simply take the difference of the appropriate expressions to get the area of a lune between two circles sharing the same chord.

Observe, in turn, that a cap cut off from a circle of radius $r$ by a chord of length $c$ is the area left over from the pie-wedge formed by slicing the circle from the endpoints of the chord to the centre of the circle when one subtracts the triangle formed by the centre of the circle and the endpoints of the course. We will call the height of this triangle $h$ and the angle formed by the two radii joining the centre of the circle to the endpoints of the chord $\alpha$, as in the diagram below.




By the Pythagorean Theorem – check out the sub-triangle isolated in the lower part of the diagram – the height of the triangle is given by $h = \sqrt{r^2 - \left(\frac{c}{2}\right)^2} = \sqrt{r^2 - \frac{c^2}{4}}$. It follows that the area of the triangle, taking the chord as its base, is $\dfrac{1}{2}ch = \dfrac{c}{2}\sqrt{r^2 - \dfrac{c^2}{4}}$.

The area of the pie-wedge has the same proportion to the area of the circle that $\alpha$ has to the angle for going (once!) all the way around the circle. If we measure angles in radians, so going around the circle all the way corresponds to an angle of $2\pi$, it follows that the area of the pie wedge is $\frac{\alpha}{2\pi} \cdot \pi r^2 = \frac{\alpha r^2}{2}$. To express this area in terms of $r$ and $c$, we need to express the angle $\alpha$ in terms of $r$ and $c$. If we consider the sub-triangle isolated in the lower part of the diagram again, we see that $\sin(\alpha/2) = \text{opposite}/\text{hypotenuse} = (c/2)/r = c/2r$. It follows that $\frac{\alpha}{2} = \arcsin\left(\frac{c}{2r}\right)$, and thus that $\alpha = 2\arcsin\left(\frac{c}{2r}\right)$. Hence the area of the pie-wedge is, in terms of $r$ and $c$, $\frac{r^2}{2} \cdot 2\arcsin\left(\frac{c}{2r}\right) = r^2 \arcsin\left(\frac{c}{2r}\right)$.

Putting these things together tells us that area of the cap cut off from a circle of radius $r$ by a chord of length $c$ is $r^2 \arcsin\left(\frac{c}{2r}\right) - \frac{c}{2}\sqrt{r^2 - \frac{c^2}{4}}$. It follows, in turn, that the area of the lune outside a circle of radius $R$ and inside a circle of radius $r$, with a common chord of length $c$, is:

$$\begin{aligned}
\text{Area lune} &= \left[r^2 \arcsin\left(\frac{c}{2r}\right) - \frac{c}{2}\sqrt{r^2 - \frac{c^2}{4}}\right] - \left[R^2 \arcsin\left(\frac{c}{2R}\right) - \frac{c}{2}\sqrt{R^2 - \frac{c^2}{4}}\right] \\
&= r^2 \arcsin\left(\frac{c}{2r}\right) - \frac{c}{2}\sqrt{r^2 - \frac{c^2}{4}} - R^2 \arcsin\left(\frac{c}{2R}\right) + \frac{c}{2}\sqrt{R^2 - \frac{c^2}{4}}
\end{aligned}$$

We can test this formula out by feeding in the quantities for the lune in question **1**, namely $r = 3$, $R = 5$, and $c = 6$, to see if we get the same answer:

$$\begin{aligned}
\text{Area lune} &= 3^2 \arcsin\left(\frac{6}{2 \cdot 3}\right) - \frac{6}{2}\sqrt{3^2 - \frac{6^2}{4}} - 5^2 \arcsin\left(\frac{6}{2 \cdot 5}\right) + \frac{6}{2}\sqrt{5^2 - \frac{6^2}{4}} \\
&= 9\arcsin(1) - 3\sqrt{9-9} - 25\arcsin\left(\frac{3}{5}\right) + 3\sqrt{25-9} \\
&= 9 \cdot \frac{\pi}{2} - 3 \cdot 0 - 25\arcsin(0.6) + 3\sqrt{16} \\
&= \frac{9\pi}{2} - 25\arcsin(0.6) + 12 \approx 10.0497
\end{aligned}$$

It's the same! Yea! :-) ■