Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2020

Solutions to Assignment #1
Gamma Function

One of the big uses of integrals in various parts of mathematics is to define functions that are otherwise difficult to nail down. For example, consider the factorial function on the non-negative integers, defined by $0! = 1$ and $(n+1)! = n! \cdot (n+1)$. (It's pretty easy to check that if $n \geq 1$ is an integer, then $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$.) The factorial function turns up in many parts of mathematics, including algebra, calculus [wait till we do series!], combinatorics, and number theory. The essentially discrete factorial function has a continuous (also differential and integrable) counterpart, which also comes up a fair bit in both applied and theoretical mathematics, namely the gamma function $\Gamma(x)$. This can be defined in a number of different ways, but the most common combines a limit and an integral:

$$\Gamma(x) = \lim_{a \to \infty} \int_0^a t^{x-1} e^{-t}\, dt$$

This definition makes sense for all $x > 0$. The expression $\lim_{a \to \infty} \int_0^a t^{x-1} e^{-t}\, dt$ is usually abbreviated a little as $\int_0^\infty t^{x-1} e^{-t}\, dt$, something we'll see more of when we do "improper integrals" (§9.7 in the textbook).

1. Verify that $\Gamma(1) = 1$. *[2]*

SOLUTION. We'll be using the substitution $u = -t$, so $du = (-1)\, dt$ and $dt = (-1)\, du$.

$$\begin{aligned}
\Gamma(1) &= \lim_{a \to \infty} \int_0^a t^{1-1} e^{-t}\, dt = \lim_{a \to \infty} \int_0^a t^0 e^{-t}\, dt = \lim_{a \to \infty} \int_0^a e^{-t}\, dt = \lim_{a \to \infty} \int_{x=0}^{x=a} e^u\, (-1)\, du \\
&= \lim_{a \to \infty} (-1)e^u \Big|_{x=0}^{x=a} = \lim_{a \to \infty} (-1)e^{-x} \Big|_0^a = \lim_{a \to \infty} \left[(-1)e^{-a} - (-1)e^{-0}\right] \\
&= \lim_{a \to \infty} \left[-\frac{1}{e^a} + 1\right] = -0 + 1 = 1 \quad \text{since } e^a \to \infty \text{ as } a \to \infty. \quad \blacksquare
\end{aligned}$$

2. Show that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. *[2]*

SOLUTION. We will first compute the definite integral $\int_0^a t^x e^{-t}\, dt$, where $x > 0$, using integration by parts with $u = t^x$ and $v' = e^{-t}$, so $u' = xt^{x-1}$ and $v = (-1)e^{-t}$. Note that x is just some constant as far as the variable in the integral, namely t, is concerned, and see the solution to **1** above for the antiderivative of e^{-t}.)

$$\begin{aligned}
\int_0^a t^x e^{-t}\, dt &= t^x e^{-t}\Big|_0^a - \int_0^a xt^{x-1}(-1)e^{-t}\, dt = \left(a^x e^{-a} - 0^x e^{-0}\right) - (-1)x \int_0^a t^{x-1} e^{-t}\, dt \\
&= a^x e^{-a} + x \int_0^a t^{x-1} e^{-t}\, dt \quad \text{(Since } 0^x = 0 \text{ if } x > 0.)
\end{aligned}$$

We plug this into the definition of $\Gamma(x+1)$ and chug away. Suppose $x > 0$:

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^{(x+1)-1}e^{-t}\,dt = \lim_{a\to\infty}\int_0^a t^{(x+1)-1}e^{-t}\,dt = \lim_{a\to\infty}\int_0^a t^x e^{-t}\,dt \\
&= \lim_{a\to\infty}\left[a^x e^{-a} + x\int_0^a t^{x-1}e^{-t}\,dt\right] = \left[\lim_{a\to\infty} a^x e^{-a}\right] + \left[\lim_{a\to\infty} x\int_0^a t^{x-1}e^{-t}\,dt\right] \\
&= 0 + x\lim_{a\to\infty}\int_0^a t^{x-1}e^{-t}\,dt \quad \text{(Since } e^{-a}\to 0 \text{ faster than } a^x\to\infty \text{ as } a\to\infty.\text{)} \\
&= x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x) \quad \blacksquare
\end{aligned}$$

3. Use **1** and **2** to show that $\Gamma(n+1) = n!$ for every integer $n \geq 0$. *[2]*

SOLUTION. Recall that the factorial function is defined for all integers $n \geq 0$ by $0! = 1$ and $(n+1)! = n! \cdot (n+1)$.

First, by the solution to **1**, we have that $\Gamma(0+1) = \Gamma(1) = 1 = 0!$, which takes care of $n = 0$.

Second, suppose that we have verified that $\Gamma(k+1) = k!$ for some particular integer $k \geq 0$. Then, by the solution to **2**, we have that $\Gamma((k+1)+1) = (k+1)\Gamma(k+1) = (k+1)\cdot k! = (k+1)!$.

The argument above is a proof by induction: the first fact is the base step of the proof, and the second fact is the inductive step of the proof. A little less formally, we have

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) = n\Gamma((n-1)+1) \\
&= n(n-1)\Gamma(n-1) = n(n-1)\Gamma((n-2)+1) \\
&= n(n-1)(n-2)\Gamma(n-2) = n(n-1)(n-2)\Gamma((n-3)+1) \\
&\ \ \vdots \\
&= n(n-1)(n-2)\cdots 3\cdot 2\cdot \Gamma(2) = n(n-1)(n-2)\cdots 3\cdot 2\cdot \Gamma(1+1) \\
&= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot \Gamma(1) = n!\cdot 1 = n!\,,
\end{aligned}$$

as required. $\blacksquare$

4. Use the limit definition of the derivative and the improper integral definition of $\Gamma(x)$ to find an integral definition of its derivative, $\Gamma'(x)$. *[4]*

SOLUTION. We will throw the limit definition of the derivative at $\Gamma(x)$ and see what we can do. At one critical step we will need to use the following fact. Suppose $t > 0$; then $t^h \to 1$ as $h \to 0$, so:

$$\begin{aligned}
&\lim_{h\to 0}\frac{t^h - 1 \to 0}{h \quad \to 0} \quad \text{and, applying l'Hôpital's Rule,} \\
&= \lim_{h\to 0}\frac{\frac{d}{dt}\left(t^h - 1\right)}{\frac{d}{dh}h} = \lim_{h\to 0}\frac{\ln(t)\cdot t^h - 0}{1} = \ln(t)\cdot\lim_{h\to 0} t^h = \ln(t)\cdot 1 = \ln(t)
\end{aligned}$$

Recall that, by definition, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. We apply this, the definition of $\Gamma(x)$, and the fact above to compute $\Gamma'(x)$, one step at a time, as far as we can:

$$\begin{aligned}
\Gamma'(x) &= \lim_{h \to 0} \frac{\Gamma(x+h) - \Gamma(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left[\lim_{a \to \infty} \int_0^a t^{(x+h)-1} e^{-t}\, dt - \lim_{a \to \infty} \int_0^a t^{x-1} e^{-t}\, dt \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\lim_{a \to \infty} \left(\int_0^a t^{(x+h)-1} e^{-t}\, dt - \int_0^a t^{x-1} e^{-t}\, dt \right) \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\lim_{a \to \infty} \int_0^a \left(t^{(x+h)-1} e^{-t} - t^{x-1} e^{-t} \right) dt \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\lim_{a \to \infty} \int_0^a \left(t^{(x+h)-1} - t^{x-1} \right) e^{-t}\, dt \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\lim_{a \to \infty} \int_0^a \left(t^{h+(x-1)} - t^{x-1} \right) e^{-t}\, dt \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\lim_{a \to \infty} \int_0^a \left(t^h - 1 \right) t^{x-1} e^{-t}\, dt \right] \\
&= \lim_{h \to 0} \left[\lim_{a \to \infty} \frac{1}{h} \int_0^a \left(t^h - 1 \right) t^{x-1} e^{-t}\, dt \right] \\
&= \lim_{h \to 0} \left[\lim_{a \to \infty} \int_0^a \left(\frac{t^h - 1}{h} \right) t^{x-1} e^{-t}\, dt \right] \\
&= \lim_{a \to \infty} \left[\lim_{h \to 0} \int_0^a \left(\frac{t^h - 1}{h} \right) t^{x-1} e^{-t}\, dt \right] \\
&= \lim_{a \to \infty} \int_0^a \left(\lim_{h \to 0} \frac{t^h - 1}{h} \right) t^{x-1} e^{-t}\, dt = \lim_{a \to \infty} \int_0^a \ln(t) t^{x-1} e^{-t}\, dt
\end{aligned}$$

The last limit-integral is very close to the definition of $\Gamma(x)$, but the $\ln(t)$ makes it different enough, and hard enough, to stop there, at least for now.

There are a couple of steps in the calculation above that one might be properly sceptical of, particularly interchanging the limits and bringing the limit as $h \to 0$ inside the definite integral. (Note that the definite integral is itself officially defined using limits.) These steps can be justified, but the proofs are largely outside the scope of this course. ■