Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2020

Solutions to the Quizzes

Quiz #1. *Thursday, 16 January [15 minutes]*

Compute each of the following integrals.

1. $\displaystyle\int x^2 \ln(x)\,dx$ *[2.5]* 2. $\displaystyle\int_0^1 \frac{1}{(1+x)\sqrt{x}}\,dx$ *[2.5]*

SOLUTION. 1. We will use integration by parts, with $u = \ln(x)$ and $v' = x^2$, so that $u' = \dfrac{1}{x}$ and $v = \dfrac{x^3}{3}$.

$$\begin{aligned}\int x^2 \ln(x)\,dx &= \int u \cdot v'\,dx = u \cdot v - \int u' \cdot v\,dx = \ln(x) \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3}\,dx \\ &= \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx = \frac{x^3 \ln(x)}{3} - \frac{1}{3} \cdot \frac{x^3}{3} + C = \frac{x^3}{3}\left(\ln(x) - \frac{1}{3}\right) + C \quad \square\end{aligned}$$

2. We will try to get rid of the square root complicating the issue with the substitution $w = \sqrt{x}$, so $x = w^2$ and $dw = \dfrac{1}{2\sqrt{x}}\,dx$, and thus $\dfrac{1}{\sqrt{x}}\,dx = 2\,dw$. We will keep the limits and change back to x before evaluating.

$$\begin{aligned}\int_0^1 \frac{1}{(1+x)\sqrt{x}}\,dx &= \int_0^1 \frac{1}{1+x} \cdot \frac{1}{\sqrt{x}}\,dx = \int_{x=0}^{x=1} \frac{1}{1+w^2}\,2\,dw = 2\arctan(w)\Big|_{x=0}^{x=1} \\ &= 2\arctan\left(\sqrt{x}\right)\Big|_0^1 = 2\arctan\left(\sqrt{1}\right) - 2\arctan\left(\sqrt{0}\right) \\ &= 2\arctan(1) - 2\arctan(0) = 2 \cdot \frac{\pi}{4} - 2 \cdot 0 = \frac{\pi}{2} \quad \blacksquare\end{aligned}$$

Quiz #2. *Thursday, 23 January [15 minutes]*

1. Evaluate $\displaystyle\int_0^1 \frac{x}{\sqrt{9x^2+16}}\,dx$. *[5]*

Solution the first. *(Trigonometric substitution.* For the sake of brevity, we will go straight to a trigonometric substitution instead of doing preliminary substitution to simplify the expression inside the square root. Since we have a "+" inside the square root, we will use a substitution of the form $x = a\tan(\theta)$, intending to exploit the identity $1+\tan^2(\theta) = \sec^2(\theta)$ to eliminate the square root. To do this, we will need to be able to factor $1+\tan^2(\theta)$ out of $9x^2+16 = 9a^2\tan^2(x)+16$, which will require $9a^2 = 16$, so that $a^2 = \frac{16}{9}$, *i.e.* $a = \frac{4}{3}$ will do.

Since $x = \frac{4}{3}\tan(\theta)$, $\tan(\theta) = \frac{3}{4}x$, $dx = \frac{4}{3}\sec^2(\theta)\,d\theta$, and $9x^2+16 = 9\left(\frac{4}{3}\right)^2\tan^2(\theta) + 16 = 16\tan^2(\theta)+16 = 16\left(\tan^2(\theta)+1\right) = 16\sec^2(\theta)$. We'll keep the old limits and substitute back before evaluating.

$$\begin{aligned}
\int_0^1 \frac{x}{\sqrt{9x^2+16}}\,dx &= \int_{x=0}^{x=1} \frac{\frac{4}{3}\tan(\theta)}{\sqrt{16\sec^2(\theta)}}\,\frac{4}{3}\sec^2(\theta)\,d\theta = \int_{x=0}^{x=1} \frac{\frac{16}{9}\tan(\theta)\sec^2(\theta)}{4\sec(\theta)}\,d\theta \\
&= \frac{4}{9}\int_{x=0}^{x=1} \tan(\theta)\sec(\theta)\,d\theta = \frac{4}{9}\sec(\theta)\Big|_{x=0}^{x=1} \qquad \text{(Because } \tfrac{d}{d\theta}\sec(\theta) = ?\text{)} \\
&= \frac{4}{9}\sqrt{1+\tan^2(\theta)}\,\Big|_{x=0}^{x=1} = \frac{4}{9}\sqrt{1+\left(\frac{3}{4}x\right)^2}\,\Bigg|_{x=0}^{x=1} \\
&= \sqrt{\frac{16}{81}+\frac{16}{81}\cdot\frac{9}{16}x^2}\,\Bigg|_{x=0}^{x=1} = \sqrt{\frac{16}{81}+\frac{1}{9}x^2}\,\Bigg|_{x=0}^{x=1} \\
&= \sqrt{\frac{16}{81}+\frac{1}{9}} - \sqrt{\frac{16}{81}} = \sqrt{\frac{16+9}{81}} - \frac{4}{9} = \frac{5}{9} - \frac{4}{9} = \frac{1}{9} \qquad \square
\end{aligned}$$

Solution the second. *(Basic substitution.)* Notice that we have an x in the numerator of the integrand. We'll use the substitution $u = 9x^2+16$, so $du = 18x\,dx$ and $x\,dx = \frac{1}{18}\,du$. We'll also change the limits as we go along: $\begin{array}{ccc} x & 0 & 1 \\ u & 16 & 25 \end{array}$.

$$\begin{aligned}
\int_0^1 \frac{x}{\sqrt{9x^2+16}}\,dx &= \int_{16}^{25} \frac{1}{\sqrt{u}}\cdot\frac{1}{18}\,du = \frac{1}{18}\int_{16}^{25} u^{-1/2}\,du = \frac{1}{18}\cdot\frac{u^{1/2}}{1/2}\Big|_{16}^{25} \\
&= \frac{1}{9}u^{1/2}\Big|_{16}^{25} = \frac{1}{9}\cdot 5 - \frac{1}{9}\cdot 4 = \frac{1}{9}
\end{aligned}$$

This is much easier than the trigonometric substitution. The moral is to keep an open mind about which technique to use: very often, more than one method to solve a given problem may be available, but one may well be a good deal easier than another. ■

Quiz #3. *Thursday, 30 January [20 minutes]*

Compute *one* (1) of the following integrals.

1. $\displaystyle\int \frac{x+1}{\sqrt{4x^2-8x}}\,dx$ *[5]* 2. $\displaystyle\int \frac{2x^2+3}{(x+1)\left(x^2+4\right)}\,dx$ *[5]*

SOLUTIONS. 1. *(Trigonometric substitution.)* Since we have a quadratic expression inside a square root, we will try to integrate using a suitable trigonometric substitution. Before we can do so, however, we must put that quadratic expression into a more useful form by completing the square:

$$4x^2-8x = 4\left(x^2-2x\right) = 4\left(x^2-2x+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)\right) = 4\left((x-1)^2-1\right)$$

We will consequently use the initial substitution $u=x-1$, so $du=dx$ and $x=u+1$, so $x+1=u+2$:

$$\int \frac{x+1}{\sqrt{4x^2-8x}}\,dx = \int \frac{x+1}{\sqrt{4\left((x-1)^2-1\right)}}\,dx = \int \frac{x+1}{2\sqrt{(x-1)^2-1}}\,dx = \frac{1}{2}\int \frac{u+2}{\sqrt{u^2-1}}\,du$$

To deal with the $\sqrt{u^2-1}$, we will use the trigonometric substitution $u=\sec(\theta)$, so $du=\sec(\theta)\tan(\theta)\,d\theta$ and $\sqrt{u^2-1}=\sqrt{\sec^2(\theta)-1}=\sqrt{\tan^2(\theta)}=\tan(\theta)$.

$$\begin{aligned}
\int \frac{x+1}{\sqrt{4x^2-8x}}\,dx &= \frac{1}{2}\int \frac{u+2}{\sqrt{u^2-1}}\,du = \frac{1}{2}\int \frac{\sec(\theta)+2}{\sqrt{\sec^2(\theta)-1}}\,\sec(\theta)\tan(\theta)\,d\theta \\
&= \frac{1}{2}\int \frac{\sec(\theta)+2}{\tan(\theta)}\,\sec(\theta)\tan(\theta)\,d\theta = \frac{1}{2}\int (\sec(\theta)+2)\sec(\theta)\,d\theta \\
&= \frac{1}{2}\int \left(\sec^2(\theta)+2\sec(\theta)\right)\,d\theta = \frac{1}{2}\int \sec^2(\theta)\,d\theta + \frac{2}{2}\int \sec(\theta)\,d\theta \\
&= \frac{1}{2}\tan(\theta) + \ln\left(\sec(\theta)+\tan(\theta)\right) + C
\end{aligned}$$

Finally, we undo the substitutions in the antiderivative:

$$\begin{aligned}
\int \frac{x+1}{\sqrt{4x^2-8x}}\,dx &= \frac{1}{2}\tan(\theta) + \ln\left(\sec(\theta)+\tan(\theta)\right) + C \\
&= \frac{1}{2}\sqrt{u^2-1} + \ln\left(u+\sqrt{u^2-1}\right) + C \\
&= \frac{1}{2}\sqrt{(x-1)^2-1} + \ln\left((x-1)+\sqrt{(x-1)^2-1}\right) + C \\
&= \frac{1}{2}\sqrt{x^2-2x} + \ln\left(x-1+\sqrt{x^2-2x}\right) + C \qquad \square
\end{aligned}$$

2. *(Integration by partial fractions.)* Since the integrand is a rational function (*i.e.* a ratio of polynomials), we will apply the technique of partial fractions. Fortunately, the denominator comes pre-factored; since $x^2+4 \geq 4 > 0$ has no roots, there is no further factoring to be done, so we can go stright to the partial fraction expansion:

$$\begin{aligned}
\frac{2x^2+3}{(x+1)\left(x^2+4\right)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+4} = \frac{A\left(x^2+4\right)+(Bx+C)(x+1)}{(x+1)\left(x^2+4\right)} \\
&= \frac{Ax^2+4A+Bx^2+Bx+Cx+C}{(x+1)\left(x^2+4\right)} \\
&= \frac{(A+B)x^2+(B+C)x+(4A+C)}{(x+1)\left(x^2+4\right)}
\end{aligned}$$

Since the first and last fractions are equal and have the same denominator, the numerators must be equal; since two polynomials are equal only if they have the same coefficients attached to the same powers, it follows that $A+B=2$, $B+C=0$, and $4A+C=3$. Adding the first and last of these equations gives $5A+B+C=5$; since $B+C=0$, it follows that $5A=5$, *i.e.* $A=1$. Since $A+B=2$, it then follows that $B=2-A=2-1=1$, and since $B+C=0$, this implies that $C=-B=-1$. Thus $\dfrac{2x^2+3}{(x+1)\left(x^2+4\right)} = \dfrac{1}{x+1}+\dfrac{x-1}{x^2+4}$. Onwards to integration!

$$\begin{aligned}
\int \frac{2x^2+3}{(x+1)\left(x^2+4\right)}\,dx &= \int \left(\frac{1}{x+1}+\frac{x-1}{x^2+4}\right)dx = \int \frac{1}{x+1}\,dx + \int \frac{x-1}{x^2+4}\,dx \\
&= \int \frac{1}{x+1}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}
\end{aligned}$$

We will use substitutions to simplify each of the three integrals on the last line above. Iin the first we will use the substitution $u=x+1$, so $du=dx$; in the second we will use the substitution $v=x^2+4$, so $dv=2x\,dx$ and $x\,dx=\frac{1}{2}\,dv$; in the third we will use the substitution $x=2w$, so $dx=2\,dw$ and $w=\dfrac{x}{2}$.

$$\begin{aligned}
\int \frac{2x^2+3}{(x+1)\left(x^2+4\right)}\,dx &= \int \frac{1}{x+1}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx \\
&= \int \frac{1}{u}\,du + \int \frac{1}{v}\cdot\frac{1}{2}\,dv - \int \frac{1}{4w^2+4}\,2dw \\
&= \ln(u) + \frac{1}{2}\ln(v) - \frac{2}{4}\int \frac{1}{w^2+1}\,dw \\
&= \ln(x+1) + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{2}\arctan(w) + C \\
&= \ln(x+1) + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C \qquad \blacksquare
\end{aligned}$$

Quiz #4. *Thursday, 6 February [20 minutes]*

1. Evaluate $\displaystyle\int_0^\infty \frac{1}{(x+1)^2(x+2)}\,dx$. *[5]*

SOLUTION. By definition, $\displaystyle\int_0^\infty \frac{1}{(x+1)^2(x+2)}\,dx = \lim_{a\to\infty}\int_0^a \frac{1}{(x+1)^2(x+2)}\,dx$. To compute the limit, we need to compute the definite integral in the limit first; to compute the definite integral, we need to find the antiderivative of the inetegrand. Since the integrand is a rational function, we apply the technique of partial fractions.

Since $\dfrac{1}{(x+1)^2(x+2)}$ is a rational function in which the degree of the numerator, 0, is less than the degree of the denominator, 3, we can proceed directly to the factoring of the denominator. Fortunately for us, we have nothing to do here, since the denominator, $(x+1)^2(x+2)$, has been presented to us fully factored into linear factors, one of which, $(x+1)^2$, occurs to the power 2. This means the partial fraction expansion of the integrand has the form:

$$\begin{aligned}
\frac{1}{(x+1)^2(x+2)} &= \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x+2} = \frac{A(x+2)+B(x+1)(x+2)+C(x+1)^2}{(x+1)^2(x+2)} \\
&= \frac{A(x+2)+B\left(x^2+3x+2\right)+C\left(x^2+2x+1\right)}{(x+1)^2(x+2)} \\
&= \frac{(B+C)x^2+(A+3B+2C)x+(2A+2B+C)}{(x+1)^2(x+2)}
\end{aligned}$$

Since their denominators are equal, the numerators in the first and last rational functions in this chain of equalities must be equal also. As two polynomials are equal only if all coefficients of equal powers are equal, it follows that we must have $B+C=0$, $A+3B+2C=0$, and $2A+2B+C=1$. The first of these three equations implies that $B=-C$; plugging this into the second equations yields $A+3(-C)+2C=A-C=0$, so $A=C$; plugging both of these into the third equation gives us $2C+2(-C)+C=C=1$. Thus $A=C=1$ and $B=-C=-1$. This means that the partial fraction expansion of the integrand is:

$$\frac{1}{(x+1)^2(x+2)} = \frac{1}{(x+1)^2} + \frac{-1}{x+1} + \frac{1}{x+2}$$

On to find the antiderivative! We will divide up the indefinite integral according to the partial fraction expansion and handle each piece separately. In the first and second pieces we will use the substitution $u=x+1$, so $du=dx$, and in the third piece we will use the substitution $w=x+2$, so $dw=dx$.

$$\begin{aligned}
\int \frac{1}{(x+1)^2(x+2)}\,dx &= \int \left(\frac{1}{(x+1)^2} + \frac{-1}{x+1} + \frac{1}{x+2}\right)dx \\
&= \int \frac{1}{(x+1)^2}\,dx - \int \frac{1}{x+1}\,dx + \int \frac{1}{x+2}\,dx \\
&= \int \frac{1}{u^2}\,du - \int \frac{1}{u}\,du + \int \frac{1}{w}\,dw = -\frac{1}{u} - \ln u + \ln(w) + C \\
&= -\frac{1}{x+1} - \ln(x+1) + \ln(x+2) + C \\
&= -\frac{1}{x+1} + \ln\left(\frac{x+2}{x+1}\right) + C
\end{aligned}$$

It now follows that:

$$\begin{aligned}
\int_0^a \frac{1}{(x+1)^2(x+2)}\,dx &= \left[-\frac{1}{x+1} + \ln\left(\frac{x+2}{x+1}\right)\right]\Big|_0^a \\
&= \left[-\frac{1}{a+1} + \ln\left(\frac{a+2}{a+1}\right)\right] - \left[-\frac{1}{0+1} + \ln\left(\frac{0+2}{0+1}\right)\right] \\
&= \left[-\frac{1}{a+1} + \ln\left(\frac{a+2}{a+1}\right)\right] - [\ln(2) - 1]
\end{aligned}$$

As a small sanity check, observe that the integral is well-defined for all $a \geq 0$, because the denominator only has $x = -1$ and $x = -2$ as roots.

We are finally ready to compute the limit and hence evaluate the improper integral:

$$\begin{aligned}
\int_0^\infty \frac{1}{(x+1)^2(x+2)}\,dx &= \lim_{a\to\infty} \int_0^a \frac{1}{(x+1)^2(x+2)}\,dx \\
&= \lim_{a\to\infty} \left(\left[-\frac{1}{a+1} + \ln\left(\frac{a+2}{a+1}\right)\right] - [\ln(2) - 1]\right) \\
&= \left[-\lim_{a\to\infty} \frac{1}{a+1}\right] + \left[\lim_{a\to\infty} \ln\left(\frac{a+2}{a+1}\right)\right] - [\ln(2) - 1] \\
&= -0 + \ln\left(\lim_{a\to\infty} \frac{a+2}{a+1}\right) - [\ln(2) - 1] \\
&= \ln\left(\lim_{a\to\infty} \frac{a+2}{a+1} \cdot \frac{\frac{1}{a}}{\frac{1}{a}}\right) - \ln(2) + 1 \\
&= \ln\left(\lim_{a\to\infty} \frac{1+\frac{2}{a}}{1+\frac{1}{a}}\right) - \ln(2) + 1 \\
&= \ln\left(\frac{1+0}{1+0}\right) - \ln(2) + 1 = \ln(1) - \ln(2) + 1 \\
&= 0 - \ln(2) + 1 = 1 - \ln(2) \approx 0.3069 \qquad \blacksquare
\end{aligned}$$

Quiz #5. *Thursday, 13 February [10 minutes]*

1. Find the arc-length of the curve $y = \frac{2}{3}x^{3/2}$, where $0 \le x \le 3$. *[5]*

SOLUTION. We'll use the arc-length formula $\displaystyle\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$. Note that in this case $\displaystyle\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2}{3}x^{3/2}\right) = \frac{2}{3} \cdot \frac{3}{2}x^{1/2} = x^{1/2} = \sqrt{x}$. Hence:

$$\text{arc-length} = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 \sqrt{1 + \left(\sqrt{x}\right)^2}\, dx = \int_0^3 \sqrt{1+x}\, dx$$

We'll substitute $u = x + 1$, so $du = dx$, and change limits: $\begin{array}{ccc} x & 0 & 3 \\ u & 1 & 4 \end{array}$.

$$= \int_1^4 \sqrt{u}\, du = \int_1^4 u^{1/2}\, du = \left.\frac{u^{3/2}}{3/2}\right|_1^4 = \left.\frac{2}{3}u^{3/2}\right|_1^4 - \frac{2}{3} \cdot 4^{3/2} = \frac{2}{3} \cdot 1^{3/2}$$

$$= \frac{2}{3} \cdot 2^3 - \frac{2}{3} \cdot 1 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}. \qquad \blacksquare$$