Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2020

Solutions to the Take-Home Final Examination

Released at noon on Tuesday, 14 April, 2020.
Due by noon on Friday, 17 April, 2020.

INSTRUCTIONS

- You may consult your notes, handouts, and textbook from this course and any other math courses you have taken or are taking now. You may also use a calculator. However, you may not consult any other source, or give or receive any other aid, except for asking the instructor to clarify instructions or questions.
- Please submit an electronic copy of your solutions, preferably as a single pdf (a scan of handwritten solutions should be fine), via the Assignment module on Blackboard. If that doesn't work, please email your solutions to the intructor.
- Do all three (3) of Parts **I** – **III**, and, if you wish, Part **IV** as well.

Part I. Do both of **1** and **2**. *[40 = 2×20 each]*

1. Compute the integrals in four (4) of **a** – **f**. *[20 = 4×5 each]*

a. $\displaystyle\int_0^{\pi/2} \cos(x)\sqrt{1+\sin^2(x)}\,dx$ **b.** $\displaystyle\int 2x^3e^{-x^2}\,dx$ **c.** $\displaystyle\int \frac{(x+1)^2}{x^2+1}\,dx$

d. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2(x)\cos^3(x)\,dx$ **e.** $\displaystyle\int_0^{\infty} xe^{-x}\,dx$ **f.** $\displaystyle\int e^x\cos(x)\,dx$

SOLUTIONS. **a.** *(Substitutions and a reduction formula.)* We will first use the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$, and change the limits as we go along: $\begin{matrix} x & 0 & \pi/2 \\ u & 0 & 1 \end{matrix}$. This gives us:

$$\int_0^{\pi/2} \cos(x)\sqrt{1+\sin^2(x)}\,dx = \int_0^1 \sqrt{1+u^2}\,du$$

We will now use the substitution $u = \tan(\theta)$, so $du = \sec^2(\theta)$, and change the limits again as we go along: $\begin{matrix} u & 0 & 1 \\ \theta & 0 & \pi/4 \end{matrix}$. Thus:

$$\begin{aligned}\int_0^{\pi/2} \cos(x)\sqrt{1+\sin^2(x)}\,dx &= \int_0^1 \sqrt{1+u^2}\,du = \int_0^{\pi/4} \sqrt{1+\tan^2(\theta)}\,\sec^2(\theta)\,d\theta \\ &= \int_0^{\pi/4} \sqrt{\sec^2(\theta)}\,\sec^2(\theta)\,d\theta = \int_0^{\pi/4} \sec^3(\theta)\,d\theta\end{aligned}$$

[Those disliking the two-stage substitution above are invited to consider trying to substitute directly, $\sin(x) = \tan(\theta)$, and see what they can make of that. It works if you are careful enough, but it's very easy to go wrong.]

At this point we will invoke the integral reduction formula for powers of $\sec(\theta)$:

$$
\begin{aligned}
\int_0^{\pi/2} \cos(x)\sqrt{1+\sin^2(x)}\,dx &= \int_0^{\pi/4} \sec^3(\theta)\,d\theta \\
&= \frac{1}{3-1}\tan(\theta)\sec^{3-2}(\theta)\Big|_0^{\pi/4} + \frac{3-2}{3-1}\int_0^{\pi/4}\sec(\theta)\,d\theta \\
&= \frac{1}{2}\tan(\theta)\sec(\theta)\Big|_0^{\pi/4} + \frac{1}{2}\ln\left(\sec(\theta)+\tan(\theta)\right)\Big|_0^{\pi/4} \\
&= \frac{1}{2}\tan\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{4}\right) - \frac{1}{2}\tan(0)\sec(0) \\
&\quad + \frac{1}{2}\ln\left(\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)\right) - \frac{1}{2}\ln\left(\tan(0)+\sec(0)\right) \\
&= \frac{1}{2}\cdot 1\cdot\sqrt{2} - \frac{1}{2}\cdot 0\cdot 1 + \frac{1}{2}\ln\left(\sqrt{2}+1\right) - \frac{1}{2}\ln(0+1) \\
&= \frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left(\sqrt{2}+1\right) \qquad \square
\end{aligned}
$$

b. *(Substitution and integration by parts.)* We will first use the substitution $w=-x^2$, so $dw=-2x\,dx$ and hence $2x\,dx=(-1)\,dw$, while $x^2=(-1)w$. Then

$$\int 2x^3e^{-x^2}\,dx = \int 2x\cdot x^2\cdot e^{-x^2}\,dx = \int(-1)we^w\,(-1)\,dw = \int we^w\,dw\,,$$

at which point we can easily use integration by parts with $u=w$ and $v'=e^w$, so $u'=1$ and $v=e^w$.

$$
\begin{aligned}
\int 2x^3e^{-x^2}\,dx &= \int we^w\,dw = we^w - \int 1e^w\,dw = we^w - e^w + C \\
&= -x^2e^{-x^2} - e^{-x^2} + C = -\left(x^2+1\right)e^{-x^2} + C \qquad \square
\end{aligned}
$$

c. *(Algebra and substitution.)* After a bit of algebra we will use the substitution $u=x^2+1$, so $du=2x\,dx$.

$$
\begin{aligned}
\int \frac{(x+1)^2}{x^2+1}\,dx &= \int \frac{x^2+2x+1}{x^2+1}\,dx = \int\left(\frac{x^2+1}{x^2+1}+\frac{2x}{x^2+1}\right)dx \\
&= \int 1\,dx + \int\frac{2x}{x^2+1}\,dx = x + \int\frac{1}{u}\,du = x+\ln(u)+C \\
&= x+\ln\left(x^2+1\right)+C
\end{aligned}
$$

One could go whole hog and use the partial fraction technology after multiplying out $(x+1)^2$, but it's quicker to do it as above. $\square$

d. *(Trig identity and substitution.)* We will use the identity $\cos^2(x) = 1 - \sin^2(x)$ to set up the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$, and change the limits as we go along, so $\begin{matrix} x & -\pi/2 & \pi/2 \\ u & -1 & 1 \end{matrix}$.

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \sin^2(x)\cos^3(x)\,dx &= \int_{-\pi/2}^{\pi/2} \sin^2(x)\cos^2(x)\cos(x)\,dx \\
&= \int_{-\pi/2}^{\pi/2} \sin^2(x)\left(1-\sin^2(x)\right)\cos(x)\,dx \\
&= \int_{-1}^{1} u^2\left(1-u^2\right)\,du = \int_{-1}^{1}\left(u^2-u^4\right)\,du \\
&= \left(\frac{u^3}{3}-\frac{u^5}{5}\right)\Big|_{-1}^{1} = \left(\frac{1^3}{3}-\frac{1^5}{5}\right) - \left(\frac{(-1)^3}{3}-\frac{(-1)^5}{5}\right) \\
&= \left(\frac{1}{3}-\frac{1}{5}\right) - \left(\frac{-1}{3}-\frac{-1}{5}\right) = \frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{5} \\
&= \frac{2}{3}-\frac{2}{5} = \frac{10}{15}-\frac{6}{15} = \frac{4}{15}
\end{aligned}$$

One could also do this one using the reduction formula for combined powers of $\sin(x)$ and $\cos(x)$. □

e. *Improper integral and integration by parts.* We will first use integration by parts to work out the necessary indefinite integral and then use that to help compute the given improper integral. The integration by parts will use $u = x$ and $v' = e^{-x}$, so $u' = 1$ and $v = (-1)e^{-x}$.

$$\begin{aligned}
\int xe^{-x}\,dx &= x(-1)e^{-x} - \int 1(-1)e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx \\
&= -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C
\end{aligned}$$

Now for the improper integral:

$$\begin{aligned}
\int_0^\infty xe^{-x}\,dx &= \lim_{a\to\infty}\int_0^a xe^{-x}\,dx = \lim_{a\to\infty} -(x+1)e^{-x}\big|_0^a \\
&= \lim_{a\to\infty}\left(\left[-(a+1)e^{-a}\right] - \left[-(1+0)e^{-0}\right]\right) = \lim_{a\to\infty}\left(\frac{-a-1}{e^a}+1\right) \\
\text{[l'Hôpital's Rule]} \quad &= 1 + \lim_{a\to\infty}\frac{-a-1}{e^a}\begin{matrix}\to -\infty \\ \to +\infty\end{matrix} = 1 + \lim_{a\to\infty}\frac{\frac{d}{da}(-a-1)}{\frac{d}{da}e^a} \\
&= 1 + \lim_{a\to\infty}\frac{-1}{e^a}\begin{matrix}\to -1 \\ \to \infty\end{matrix} = 1 + 0 = 1 \qquad \square
\end{aligned}$$

f. *(Integration by parts and some algebra.)* We will first use integration by parts, with $u = e^x$ and $v' = \cos(x)$, so $u' = e^x$ and $v = \sin(x)$.

$$\int e^x\cos(x)\,dx = e^x\sin(x) - \int e^x\sin(x)\,dx$$

We will use integration by parts again, this time with $s = e^x$ and $t' = \sin(x)$, so $s' = e^x$ and $t = -\cos(x)$, to handle the remaining integral.

$$\begin{aligned}\int e^x \cos(x)\, dx &= e^x \sin(x) - \int e^x \sin(x)\, dx \\ &= e^x \sin(x) - \left[e^x \left(-\cos(x)\right) - \int e^x \left(-\cos(x)\right)\, dx \right] \\ &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\, dx\end{aligned}$$

Comparing the beginning and the end of this sequence of equations, we go on to solve for $\int e^x \cos(x)\, dx$:

$$\begin{aligned}\int e^x \cos(x)\, dx &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\, dx \\ \implies 2 \int e^x \cos(x)\, dx &= e^x \sin(x) + e^x \cos(x) \\ \implies \int e^x \cos(x)\, dx &= \frac{1}{2} \left(e^x \sin(x) + e^x \cos(x)\right) + C\end{aligned}$$

We belatedly remember the constant of integration at the last step ... :-) ■

2. Determine whether the series converges or not in four (4) of **a** – **f**. *[20 = 4×5 each]*

a. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{2^n n^n}$ **b.** $\displaystyle\sum_{n=0}^{\infty} 3^n 2^{-n}$ **c.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{e^n + n}$

d. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ **e.** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\left[\ln(n)\right]^2}$ **f.** $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{e^{2n} + 1}$

SOLUTIONS. **a.** *(Root Test)* Since the individual terms are built out of nth powers using only multiplication and division, we try the Root Test.

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{e^n}{2^n n^n}\right|^{1/n} = \lim_{n\to\infty} \left|\left(\frac{e}{2n}\right)^n\right|^{1/n} = \lim_{n\to\infty} \frac{e}{2n} \begin{matrix}\to e \\ \to \infty\end{matrix} = 0$$

Since the limit exists and is less than 1, the Root Test tells us that the series converges absolutely. □

b. *(Geometric series)* Since $\displaystyle\sum_{n=0}^{\infty} 3^n 2^{-n} = \sum_{n=0}^{\infty} \frac{3^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ is a geometric series with common ratio $r = \dfrac{3}{2} > 1$, it diverges.

The Root and Ratio Tests would also solve this problem very quickly. □

c. *(Basic Comparison Test)* Note that $0 \leq \left|\dfrac{(-1)^n}{e^n+n}\right| = \dfrac{1}{e^n+n} \leq \dfrac{1}{e^n}$ for all $n \geq 0$ because $e^n + n \geq e^n$ when $n \geq 0$. Since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ is a geometric series with common ratio $|r| = \dfrac{1}{e} < 1$, it converges. It follows by the (Basic) Comparison Test that the given series converges absolutely, and hence converges.

The Limit Comparison and Alternating Series Tests would also work to solve this problem, with increasing amounts of overall effort. □

d. *(Basic Comparison Test)* Since $0 < \ln(n) < n$ for all $n \geq 2$, we have $0 \leq \dfrac{1}{n} < \dfrac{1}{\ln(n)}$ for all $n \geq 2$. Since the harmonic series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is known to diverge (shown in class, or you could use the p-Test or the Integral Test), it follows by the (Basic) Comparison Test that the given series diverges.

The Limit Comparison Test would also work here, albeit with a little more effort. □

e. *(Integral Test)* Passes the Divergence Test, which only means it *might* converge; isn't an alternating series, so the Alternating Series Test doesn't apply; isn't a series of terms that are rational functions of n, so neither p-Test applies; Comparison Tests might work, if one could think of what to compare it to; no nth powers, so the Root Test would be hard to use; and it's hard to deal with the limit the Ratio Test requires. What to do? By process of elimination, we try the Integral Test.

We will use the substitution $u = \ln(x)$, so $du = \dfrac{1}{x}\,dx$, to work out the definite integral inside the limit below.

$$\begin{aligned}\int_3^{\infty} \frac{1}{x\,[\ln(x)]}\,dx &= \lim_{a\to\infty} \int_3^{a} \frac{1}{x\,[\ln(x)]}\,dx = \lim_{a\to\infty} \int_{x=3}^{x=a} \frac{1}{u^2}\,du = \lim_{a\to\infty} \int_{x=3}^{x=a} u^{-2}\,du \\ &= \lim_{a\to\infty} \left.\frac{u^{-1}}{-1}\right|_{x=3}^{x=a} = \lim_{a\to\infty} \left.\frac{-1}{u}\right|_{x=3}^{x=a} = \lim_{a\to\infty} \left(\frac{-1}{a} - \frac{-1}{3}\right) = 0 + \frac{1}{3} = \frac{1}{3}\end{aligned}$$

Since the corresponding improper integral converges, the Integral Test tells us that the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\,[\ln(n)]^2}$ does so as well. □

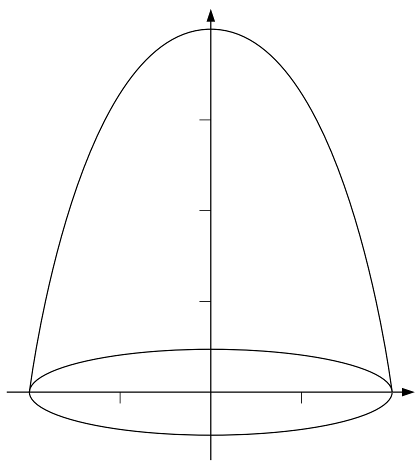
f. *(Basic Comparison Test)* We have $0 < \dfrac{e^n}{e^{2n}+1} < \dfrac{e^n}{e^{2n}} = \dfrac{1}{e^n}$ for all $n \geq 0$ because $e^{2n} + 1 > e^{2n}$. Since the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ is a geometric series with common ration $|r| = \dfrac{1}{e} < 1$, it converges. It follows by the (Basic) Comparison Test that the series $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{e^{2n}+1}$ converges as well.

This problem could also be done using the Limit Comparison, Integral, Ratio, or even the Root Test, with varying degrees of difficulty in computing the relevant limits. ■

Part II. Do any two (2) of **3** – **5**. *[20 = 2×10 each]*

3. Find the volume of the solid obtained by revolving the region below $y = 4 - x^2$ and above $y = 0$, for $0 \leq x \leq 2$, about the y-axis. *[10]*

SOLUTION. Here is a crude sketch of the solid:

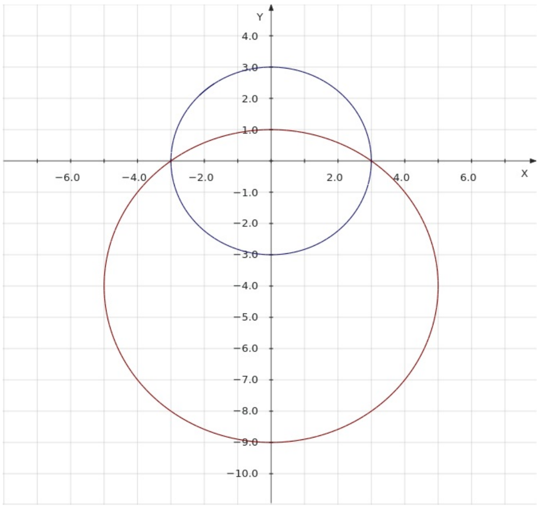


We will use the method of cylindrical shells to compute the volume of the solid. (Doing so using the disk method is about equally easy.) The shell at x, for an x with $0 \leq x \leq 2$, has radius $r = x - 0 = x$ and height $h = \left(4 - x^2\right) - 0 = 4 - x^2$. We will use x as the variable of integration because the shells are perpendicular to the x-axis. It follows that the volume of the solid is given by:

$$\begin{aligned} V &= \int_0^2 2\pi r h \, dx = \int_0^2 2\pi x \left(4 - x^2\right) \, dx = 2\pi \int_0^2 \left(4x - x^3\right) \, dx \\ &= 2\pi \left(2x^2 - \frac{x^4}{4}\right)\Big|_0^2 = 2\pi \left(2 \cdot 2^2 - \frac{2^4}{4}\right) - 2\pi \left(2 \cdot 0^2 - \frac{0^4}{4}\right) \\ &= 2\pi (8 - 4) - 0 = 8\pi \qquad \square \end{aligned}$$

4. Find the centroid of the region outside the circle $x^2 + (y+4)^2 = 25$ and inside the circle $x^2 + y^2 = 9$. *[10]*

SOLUTION. Here is a plot of the two circles:



If this seems familiar, this is basically the same picture as for question **1** on Assignment #2, except that the larger circle has been reflected in the x-axis.

First, since the region in question (the lune inside the smaller circle and outside the larger one) is symmetric about the y-axis, the centroid must be on the y-axis, *i.e.* $\overline{x} = 0$.

Second, the "mass", *i.e.* area, of the lune is the same as the area of the (symmetric) lune in question **1** of Assignment #2, namely $M = 2 + \frac{9\pi}{2} - 25 \arcsin\left(\frac{3}{5}\right) \approx 10.0497$. (Recall that we may use the handouts in this course, which includes the solutions to Assignment #2.)

Third, we need to compute the moment of the region about the x-axis, denoted in the textbook by M_x. By definition, $M_x = \int_0^3 y \cdot [\text{length of the cross-section at } y]\, dy$. Our problem is that the cross-sections of the region for $0 \le y \le 1$ come in two pieces and the cross-sections for $1 \le y \le 3$ come in one piece. This means that we have to break up the integral into two pieces.

$$
\begin{aligned}
M_x &= \int_0^3 y \cdot [\text{length of the cross-section at } y]\, dy \\
&= \int_0^1 y \cdot \left[\left(-\sqrt{25-(y+4)^2} - \left(-\sqrt{9-y^2}\right)\right) + \left(\sqrt{9-y^2} - \sqrt{25-(y+4)^2}\right)\right] dy \\
&\qquad + \int_1^3 y \cdot \left[\sqrt{9-y^2} - \left(-\sqrt{9-y^2}\right)\right] dy \\
&= \int_0^1 y \cdot 2\left[\sqrt{9-y^2} - \sqrt{25-(y+4)^2}\right] dy + \int_1^3 y \cdot 2\sqrt{9-y^2}\, dy \\
&= \int_0^1 2y\sqrt{9-y^2}\, dy - \int_0^1 2y\sqrt{25-(y+4)^2}\, dy + \int_1^3 2y\sqrt{9-y^2}\, dy \\
&= \int_0^3 2y\sqrt{9-y^2}\, dy - \int_0^1 2y\sqrt{25-(y+4)^2}\, dy
\end{aligned}
$$

In the first of the two integrals remaining we will use the substitution $u = 9 - y^2$, so $du = -2y\,dy$ and $2y\,dy = (-1)\,du$, and change the variables as we go along: $\begin{matrix} y & 0 & 3 \\ u & 9 & 0 \end{matrix}$ This gives:

$$\begin{aligned}\int_0^3 2y\sqrt{9-y^2}\,dy &= \int_9^0 \sqrt{u}\,(-1)\,du = \int_0^9 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_0^9 \\ &= \frac{2}{3}9^{3/2} - \frac{2}{3}0^{3/2} = \frac{2}{3}\cdot 27 - 0 = 18\end{aligned}$$

In the second of the two integrals we will first use the substitution $w = y + 4$, so $dw = dy$ and $y = w - 4$, and change the limits as we go along: $\begin{matrix} y & 0 & 1 \\ w & 4 & 5 \end{matrix}$. This gives:

$$\begin{aligned}\int_0^1 2y\sqrt{25-(y+4)^2}\,dy &= \int_4^5 2(w-4)\sqrt{25-w^2}\,dw \\ &= \int_4^5 2w\sqrt{25-w^2}\,dw - \int_4^5 8\sqrt{25-w^2}\,dw\end{aligned}$$

For the first of these we will use the substitution $s = 25 - w^2$, so $ds = -2w\,dw$ and $2w\,dw = (-1)\,ds$, and change the variables as we go along, $\begin{matrix} w & 4 & 5 \\ s & 9 & 0 \end{matrix}$. For the second we will use the trigonometric substitution $w = 5\sin(\theta)$, so $dw = 5\cos(\theta)\,d\theta$, but keep the limits and substitute back before evaluating. Thus:

$$\begin{aligned}\int_0^1 2y\sqrt{25-(y+4)^2}\,dy &= \int_4^5 2w\sqrt{25-w^2}\,dw - \int_4^5 8\sqrt{25-w^2}\,dw \\ &= \int_9^0 \sqrt{s}\,(-1)\,ds - 8\int_{w=4}^{w=5} \sqrt{25-25\sin^2(\theta)}\,5\cos(\theta)\,d\theta \\ &= \int_0^9 s^{1/2}\,ds - 8\int_{w=4}^{w=5} 25\cos^2(\theta)\,d\theta\end{aligned}$$

We will use the reduction formula for integrating powers of $\cos(\theta)$ to deal with the latter integral.

$$\begin{aligned}
\int_0^1 2y\sqrt{25-(y+4)^2}\,dy &= \int_0^9 s^{1/2}\,ds - 8\int_{w=4}^{w=5} 25\cos^2(\theta)\,d\theta \\
&= \frac{2}{3}s^{3/2}\Big|_0^9 - 200\left[\frac{1}{2}\cos^{2-1}(\theta)\sin(\theta) + \frac{1}{2}\int \cos^{2-2}(\theta)\,d\theta\right]_{w=4}^{w=5} \\
&= \frac{2}{3}9^{3/2} - \frac{2}{3}0^{3/2} - 200\left[\frac{1}{2}\cos(\theta)\sin(\theta) + \frac{1}{2}\int 1\,d\theta\right]_{w=4}^{w=5} \\
&= \frac{2}{3}\cdot 27 - 0 - 200\left[\frac{1}{2}\cos(\theta)\sin(\theta) + \frac{1}{2}\theta\right]_{w=4}^{w=5} \\
&= 18 - \left[100\frac{w}{5}\sqrt{1-\frac{w^2}{25}} + 100\arcsin\left(\frac{w}{5}\right)\right]_4^5 \\
&= 18 - \left[4w\sqrt{25-w^2} + 100\arcsin\left(\frac{w}{5}\right)\right]_4^5 \\
&= 18 - [(4\cdot 5\cdot 0 + 100\arcsin(1)) - (4\cdot 4\cdot 3 + 100\arcsin(0.8))] \\
&\approx 18 - \left[\left(0 + 100\frac{\pi}{2}\right) - (48 + 100\cdot 0.927295)\right] \\
&\approx 18 - [157.0796 - 140.7295] \approx 18 - 16.3501 \approx 1.6499
\end{aligned}$$

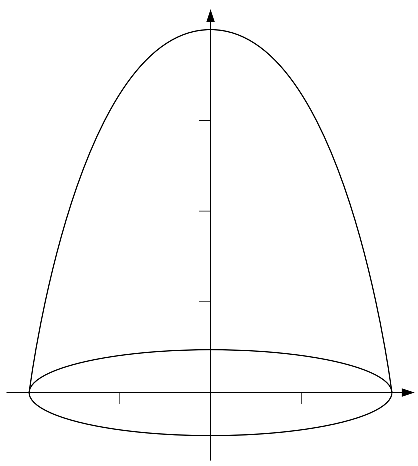
It follows that:

$$M_x = \int_0^3 2y\sqrt{9-y^2}\,dy - \int_0^1 2y\sqrt{25-(y+4)^2}\,dy \approx 18 - 1.6499 \approx 16.3501$$

Fourth, by definition, $\overline{y} = \dfrac{M_x}{M} \approx \dfrac{16.3501}{10.0497} \approx 1.6269$.
Thus the centroid of the region is at $(\overline{x},\overline{y}) \approx (0, 1.6269)$. Whew! $\square$

5. Find the area of the surface obtained by revolving the curve $y = 4 - x^2$, for $0 \leq x \leq 2$, about the y-axis. *[10]*

SOLUTION. The sketch used in the solution to **3** works here too, so here it is again:



We will plug $y = 4 - x^2$, for $0 \leq x \leq 2$, into the formula for surface area, $\int_a^b 2\pi r ds$, and integrate away.

Since $\frac{dy}{dx} = \frac{d}{dx}\left(4 - x^2\right) = -2x$, we have

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + (-2x)^2} = \sqrt{1 + 4x^2}\, dx\,,$$

and since we are revolving about the y-axis, the point at (x, y) on the curve is revolved on a circle of radius $r = x - 0 = x$. Thus the surface area of the resulting surface of revolution is given by $\int_0^2 2\pi x\sqrt{1 + 4x^2}\, dx$.

We will use the substitution $u = 1 + 4x^2$, so $du = 8x\, dx$ and hence $2x\, dx = \frac{1}{4}\, du$, and change the limits as we go along: $\begin{matrix} x & 0 & 2 \\ u & 1 & 17 \end{matrix}$

$$\begin{aligned} SA &= \int_0^2 2\pi x\sqrt{1 + 4x^2}\, dx = \int_1^{17} \pi\sqrt{u}\,\frac{1}{4}\, dx = \frac{\pi}{4}\int_1^{17} u^{1/2}\, du \\ &= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2}\Big|_1^{17} = \frac{\pi}{6} 17^{3/2} - \frac{\pi}{6} 1^{3/2} = \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \end{aligned}$$

Not a pretty answer, but there it is ... $\square$

Part III. Do any two (2) of **6** – **8**. *[20 = 2×10 each]*

6. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$. What function is it the Taylor series of? *[10]*

SOLUTION. As usual, we use the Ratio Test to find the radius of convergence.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{2(n+1)}}{((2(n+1))!}}{\frac{x^{2n}}{(2n)!}}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{(2n+2)(2n+1)}\right|$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} \begin{matrix}\to x^2 \\ \to \infty\end{matrix} = 0$$

Since $0 < 1$, the Ratio Test tells us that the given series converges absolutely for all x, *i.e.* the radius of convergence is $r = \infty$ and so its interval of convergence is $(-\infty, \infty)$.

Now, what is the function $f(x)$ which has the given series as its Taylor series? (At $x = 0$, obviously.) The given series is

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots,$$

which looks somewhat like the Taylor series of e^x, which we have seen before, namely

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

In fact the given series consists of exactly the even-numbered terms of the Taylor series of e^x. So one could guess that our unknown function $f(x)$ is somehow related to e^x.

We have seen several functions closely related to e^x – as in made from it – in this course. The one we have seen most frequently in various examples and questions is e^{-x}. It's Taylor series at 0 is just the Taylor series of e^x with $-x = (-1)x$ plugged in for x:

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

This is just the alternating version of the Taylor series of e^x. Notice that all the even-numbered terms are positive and the odd-numbered terms are negative. This means that when we add the series for e^x and e^{-x} together, the odd-numbered terms will cancel out and the even-numbered will add to themselves:

$$\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) + \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}\right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$+ 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots = 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \cdots$$

$$= 2\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Dividing by 2, we can summarize this as $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{1}{2}\left[\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) + \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}\right)\right]$. Since the right-hand side must be the Taylor series of $\dfrac{e^x + e^{-x}}{2}$ at 0, so is the left-hand side. We have seen the function $\dfrac{e^x + e^{-x}}{2}$ by the name of $\cosh(x)$ a time or three.

Thus the given series, $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, is the Taylor series at 0 of $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. ■

7. Suppose $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ is a polynomial of degree k. Find the Taylor series of $p(x)$, and find its radius and interval of convergence. *[10]*

Solution. Our polynomial can be thought of as a power series in a pretty easy way:

$$\begin{aligned} p(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_kx^k \\ &= a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + 0x^{k+1} + 0x^{k+2} + \cdots \\ &= a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + \sum_{n=k+1}^{\infty} 0x^n\,, \end{aligned}$$

Since a function equal to a power series has that power series as its Taylor series, it follows that $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + \displaystyle\sum_{n=k+1}^{\infty} 0x^n$ is its own Taylor series at 0. Because any polynomial $p(x)$ is defined (and continuous, and differentiable, and integrable :-) for all x, it has radius of convergence $r = \infty$ and interval of convergence $(-\infty, \infty)$. ■

8. Use the Taylor series of the three functions involved to show that $e^{ix} = \cos(x) + i\sin(x)$, where $i^2 = -1$, *i.e.* $i = \sqrt{-1}$. *[10]*

Solution. The Taylor series at 0 of e^x, $\cos(x)$, and $\sin(s)$ are $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, and $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, respectively. (You have probably seen these before, but they're pretty easy to work out using Taylor's formula if you haven't.) All three series converge for all x and, like most reasonable functions, converge to the functions that they are the Taylor series of.

Note that because $i^2 = -1$, we have $i^3 = -i$, $i^4 = (-1)^2 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, and so on. It follows that:

$$
\begin{aligned}
e^{ix} &= \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots \\
&= 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} - \cdots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right) + i\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) = \cos(x) + i\sin(x) \qquad \blacksquare
\end{aligned}
$$

[Total = 80]

Part IV. Bonus! If you want to, do one or both of the following problems.

41. Write a poem touching on calculus or mathematics in general. *[1]*

SOLUTION. You're on your own here! ■

42. Answer the riddle below, which supposedly gives the length of the Hellenistic mathematician Diophantus of Alexandria's life. *[1]*

126.—ΑΛΛΟ

Οὗτός τοι Διόφαντον ἔχει τάφος· ἆ μέγα θαῦμα·
καὶ τάφος ἐκ τέχνης μέτρα βίοιο λέγει.
ἕκτην κουρίζειν βιότου θεὸς ὤπασε μοίρην·
δωδεκάτην δ' ἐπιθείς, μῆλα πόρεν χνοάειν·
τῇ δ' ἄρ' ἐφ' ἑβδομάτῃ τὸ γαμήλιον ἥψατο φέγγος,
ἐκ δὲ γάμων πέμπτῳ παῖδ' ἐπένευσεν ἔτει.
αἰαῖ, τηλύγετον δειλὸν τέκος, ἥμισυ πατρὸς
†τοῦδε καὶ ἡ κρυερὸς μέτρον ἑλὼν βιότου.
πένθος δ' αὖ πισύρεσσι παρηγορέων ἐνιαυτοῖς
τῇδε πόσου σοφίῃ τέρμ' ἐπέρησε βίου.

126

This tomb holds Diophantus. Ah, how great a marvel! the tomb tells scientifically the measure of his life. God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, he clothed his cheeks with down; He lit him the light of wedlock after a seventh part, and five years after his marriage He granted him a son. Alas! late-born wretched child; after attaining the measure of half his father's life, chill Fate took him. After consoling his grief by this science of numbers for four years he ended his life.

Metrodorus, Epigram 126, Greek Anthology

SOLUTION. The answer is 84. Not because it's the answer to life, the universe, and everything, twice over ... :-) ■

THANK YOU ALL FOR BEARING WITH THE COURSE
UNDER DIFFICULT CIRCUMSTANCES. IT HAS BEEN
BOTH A PLEASURE AND AN HONOUR TO TEACH YOU.
MAY YOU AND YOURS BE WELL AND SAFE, AND
MAY WE SEE EACH OTHER AGAIN IN BETTER TIMES.