Trent University, Winter 2019

MATH 1120H Test

Friday, 1 March

Time: 11:00–11:50
Space: GCS 114

Name: Solutions

Student Number: 7481011

Question	Mark
1	____
2	____
3	____
Total	____ /30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and (all sides of) an aid sheet.

1. Compute any *four* (4) of integrals **a**–**f**. *[12 = 4 × 3 each]*

a. $\displaystyle\int_0^{\pi/2} \cos(x)\sin^3(x)\,dx$ **b.** $\displaystyle\int_2^{\infty} \frac{1}{y^3}\,dy$ **c.** $\displaystyle\int e^z\cos(z)\,dz$

d. $\displaystyle\int \frac{5}{t^2+t-6}\,dt$ **e.** $\displaystyle\int \frac{1}{\sqrt{1-9s^2}}\,ds$ **f.** $\displaystyle\int_0^1 \frac{r+1}{r^2+1}\,dr$

SOLUTIONS. **a.** We will use the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$, and change the limits as we go along: $\begin{matrix} x & 0 & \pi/2 \\ u & 0 & 1 \end{matrix}$

$$\int_0^{\pi/2} \cos(x)\sin^3(x)\,dx = \int_0^1 u^3\,du = \left.\frac{u^4}{4}\right|_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4} \quad \square$$

b. This is an improper integral, so there is a limit:

$$\begin{aligned}\int_2^{\infty} \frac{1}{y^3}\,dy &= \lim_{t\to\infty}\int_2^t y^{-3}\,dy = \lim_{t\to\infty}\left.\frac{y^{-2}}{-2}\right|_2^t = \lim_{t\to\infty}\left.\frac{-1}{2y^2}\right|_2^t \\ &= \lim_{t\to\infty}\left(\frac{-1}{2t^2} - \frac{-1}{2(-2)^2}\right) = 0 - \frac{-1}{8} = \frac{1}{8} \quad \square\end{aligned}$$

c. We will use integration by parts twice and then solve for the integral. The first time the parts will be $u = e^z$ and $v' = \cos(z)$, so $u' = e^z$ and $v = \sin(z)$; the second time they will be $s = e^z$ and $t' = \sin(z)$, so $s' = e^z$ and $t = -\cos(z)$.

$$\begin{aligned}\int e^z\cos(z)\,dz &= e^z\sin(z) - \int e^z\sin(z)\,dz \\ &= e^z\sin(z) - \left[e^z(-\cos(z)) - \int e^z(-\cos(z))\,dz\right] \\ &= e^z\sin(z) + e^z\cos(z) - \int e^z\cos(z)\,dz \\ \implies 2\int e^z\cos(z)\,dz &= e^z\sin(z) + e^z\cos(z) = e^z(\sin(z)+\cos(z)) \\ \implies \int e^z\cos(z)\,dz &= \frac{1}{2}e^z(\sin(z)+\cos(z)) + C \quad \square\end{aligned}$$

d. First, observe that the degree of the numerator in $\dfrac{5}{t^2+t-6}$ is less than the degree of the denominator, so there is no need to do long division.

Second, we factor the denominator: $t^2+t-6 = (t-2)(t+3)$. If one didn't just spot this, it is easy enough to find the roots of $t^2+t-6=0$ using the quadratic formula:

$$t = \frac{-1\pm\sqrt{1^2-4\cdot1\cdot(-6)}}{2\cdot1} = \frac{-1\pm\sqrt{25}}{2} = \frac{-1\pm5}{2} = \begin{cases}\frac{-1+5}{2} \\ \frac{-1-5}{2}\end{cases} = \begin{cases}2 \\ -3\end{cases}$$

It then follows that $t^2+t-6 = (t-2)(t-(-3)) = (t-2)(t+3)$.

Third, it follows that $\dfrac{5}{t^2+t-6} = \dfrac{A}{t-2} + \dfrac{B}{t+3}$ for some unknown constants A and B. Comparing coefficients in the numerators in

$$\frac{5}{t^2+t-6} = \frac{A}{t-2} + \frac{B}{t+3} = \frac{A(t+3)+B(t-2)}{(t-2)(t+3)} = \frac{(A+B)t+(3A-2B)}{t^2+t-6}$$

implies that $A+B=0$ and $3A-2B=5$. The first equation tells us that $B=-A$; plugging this into the second tells us that $5 = 3A-2B = 3A-2(-A) = 5A$, so we must have $A=1$ and hence also $B=-A=-1$. Thus $\dfrac{5}{t^2+t-6} = \dfrac{1}{t-2} + \dfrac{-1}{t+3}$.

Finally, we integrate:

$$\begin{aligned} \int \frac{5}{t^2+t-6}\,dt &= \int \frac{1}{t-2}\,dt + \int \frac{-1}{t+3}\,dt \qquad \text{Substitute } u=t-2, \text{ so } du=dt, \text{ and } w=t+3, \text{ so } dw=dt, \text{ respectively.} \\ &= \int \frac{1}{u}\,du - \int \frac{1}{w}\,dw = \ln(u)-\ln(w)+C \\ &= \ln(t-2)-\ln(t+3)+C \quad \square \end{aligned}$$

e. We will use the substitution $s=\dfrac{u}{3}$, so $ds=\dfrac{1}{3}\,du$, to get rid of that factor of 9, and then follow up with the trigonometric substitution $u=\sin(\theta)$, so $du=\cos(\theta)\,d\theta$. (Of course, we could also do this all in one go by substituting $s=\dfrac{1}{3}\sin(\theta)$, so $ds=\dfrac{1}{3}\cos(\theta)\,d\theta$.)

$$\begin{aligned} \int \frac{1}{\sqrt{1-9s^2}}\,ds &= \int \frac{1}{\sqrt{1-9\frac{u^2}{9}}}\cdot\frac{1}{3}\,du = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\int \frac{1}{\sqrt{1-\sin^2(\theta)}}\cos(\theta)\,d\theta \\ &= \frac{1}{3}\int \frac{\cos(\theta)}{\sqrt{\cos^2(\theta)}}\,d\theta = \frac{1}{3}\int \frac{\cos(\theta)}{\cos(\theta)}\,d\theta = \frac{1}{3}\int 1\,d\theta = \frac{1}{3}\theta + C \\ &= \frac{1}{3}\arcsin(u)+C = \frac{1}{3}\arcsin(3s)+C \quad \square \end{aligned}$$

f. Chop up and conquer! In the first piece, we will use the substitution $w=r^2+1$, so $dw=2\,dr$ and thus $dr=\dfrac{1}{2}\,dw$.

$$\begin{aligned} \int_0^1 \frac{r+1}{r^2+1}\,dr &= \int_0^1 \frac{r}{r^2+1}\,dr + \int_0^1 \frac{1}{r^2+1}\,dr = \int_{r=0}^{r=1} \frac{1}{w}\,\frac{1}{2}\,dw + \arctan(r)\big|_0^1 \\ &= \ln(w)\big|_{r=0}^{r=1} + \arctan(1)-\arctan(0) = \ln\left(r^2+1\right)\Big|_0^1 + \frac{\pi}{4} - 0 \\ &= \ln\left(1^2+1\right) - \ln\left(0^2+1\right) + \frac{\pi}{4} = \ln(2)-\ln(1)+\frac{\pi}{4} = \ln(2)-0+\frac{\pi}{4} \\ &= \ln(2)+\frac{\pi}{4} \quad \blacksquare \end{aligned}$$

2. Do any *two* (2) of parts **a**–**c**. *[8 = 2 × 4 each]*

a. Use a Right-Hand Rule sum to compute $\displaystyle\int_0^4 x\,dx$.

b. Find the area of the finite region above $y = e^x$ and below $y = (e-1)x + 1$.

c. Find the arc-length of the curve $y = \dfrac{4x}{3}$, where $0 \le x \le 3$.

SOLUTIONS. **a.** We plug the pieces of the given integral into the Right-Hand Rule formula $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{\infty} f\left(a + i\frac{b-a}{n}\right)$, using the formula $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ along the way.

$$\begin{aligned}
\int_0^4 x\,dx &= \lim_{n\to\infty} \frac{4-0}{n} \sum_{i=1}^{n} 4 \cdot \left(0 + i\frac{4-0}{n}\right) = \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{\infty} 4 \cdot i \cdot \frac{4}{n} \\
&= \lim_{n\to\infty} \frac{4}{n} \cdot 4 \cdot \frac{4}{n} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{64}{n^2} \cdot \frac{n(n+1)}{2} = \lim_{n\to\infty} 32 \cdot \frac{n+1}{n} \\
&\lim_{n\to\infty} 32\left(\frac{n}{n} + \frac{1}{n}\right) = \lim_{n\to\infty} 32\left(1 + \frac{1}{n}\right) = 32(1+0) = 32 \quad \square
\end{aligned}$$

b. Note first that $e^x = (e-1)x + 1$ when $x = 0$, since $e^0 = 1 = (e-1)0 + 1$, and when $x = 1$, since $e^1 = e = e - 1 + 1 = (e-1)1 + 1$, and that $e^x \le (e-1)x + 1$ for $0 \le x \le 1$. It follows that the region in question has area given by:

$$\begin{aligned}
A &= \int_0^1 \left((e-1)x + 1 - e^x\right)\,dx = (e-1)\frac{x^2}{2} + x - e^x \Big|_0^1 \\
&= \left[(e-1)\frac{1^2}{2} + 1 - e^1\right] - \left[(e-1)\frac{0^2}{2} + 0 - e^0\right] \\
&= \left[\frac{1}{2}e - \frac{1}{2} + 1 - e\right] - [0-1] = -\frac{1}{2}e + \frac{3}{2} = \frac{1}{2}(3-e) \quad \square
\end{aligned}$$

c. *(With calculus.)* We plug the curve into the arc-length formula $\displaystyle\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ and integrate away:

$$\begin{aligned}
\text{arc-length} &= \int_0^3 \sqrt{1 + \left(\frac{d}{dx}\left[\frac{4x}{3}\right]\right)^2}\,dx = \int_0^3 \sqrt{1 + \left(\frac{4}{3}\right)^2}\,dx = \int_0^3 \sqrt{\frac{9}{9} + \frac{16}{9}}\,dx \\
&= \int_0^3 \sqrt{\frac{25}{9}}\,dx = \int_0^3 \frac{5}{3}\,dx = \frac{5}{3}x\Big|_0^3 = \frac{5}{3}\cdot 3 - \frac{5}{3}\cdot 0 = 5 - 0 = 5 \quad \square
\end{aligned}$$

c. *(Without calculus.)* $y = \dfrac{4x}{3}$, where $0 \le x \le 3$, is the line segment joining $(0,0)$ to $(3,4)$, and is the hypotenuse of the right triangle whose third vertex is $(3,0)$. The short sides of the triangle, from $(0,0)$ to $(0,3)$ and from $(3,0)$ to $(3,4)$, have lengths 3 and 4, respectively, so the hypotenuse has length $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$ by the Pythagorean Theorem. ■

3. Do either *one* (1) of parts **a** or **b**. *[10]*

a. Compute $\displaystyle\int \frac{x^4+x^2+1}{x^3+x}\,dx$.

b. A triangular flat plate of constant thickness and density has its vertices at the points $(0,0)$, $(0,4)$, and $(4,0)$. Find the coordinates of its centroid. (You may assume that units have been chosen so that mass per unit area equals 1.)

SOLUTIONS. **a.** First, the degree of the numerator is greater than the degree of the denominator, so we need to divide the denominator into the numerator as far as we can. Fortunately, this is pretty easy to do in this case by inspection, $x^4+x^2+1 = x\left(x^3+x\right)+1$, so there is no need to resort to long division. (Not that doing so would take long in this case.) It follows that

$$\frac{x^4+x^2+1}{x^3+x} = \frac{x\left(x^3+x\right)+1}{x^3+x} = \frac{x\left(x^3+x\right)}{x^3+x} + \frac{1}{x^3+x} = x + \frac{1}{x^3+x}\,,$$

and the degree of the numerator in the remaining fraction is indeed less than the degree of the denominator.

Second, we need to fully factor the denominator x^3+x, which, again fortunately*, is easy to do $x^3+x = x\left(x^2+1\right)$. The quadratic factor x^2+1 is obviously never 0 since it is at least 1 for all x, so it has no roots and hence is an irreducible quadratic.

Third, we need to decompose the remaining fraction into "partial fractions":

$$\frac{1}{x^3+x} = \frac{1}{x\left(x^2+1\right)} = \frac{Ax+B}{x^2+1} + \frac{C}{x} = \frac{(Ax+B)x + C\left(x^2+1\right)}{x\left(x^2+1\right)} = \frac{(A+C)x^2+Bx+C}{x\left(x^2+1\right)}$$

Comparing coefficients in the numerators on the left and right ends, we see that we must have $A+C=0$, $B=0$, and $C=1$. Combining the first and last equations, we see that $A=-C=-1$, so $\dfrac{1}{x^3+x} = \dfrac{-x}{x^2+1} + \dfrac{1}{x}$.

Finally, we integrate. Along the way, we will use the substitution $u = x^2+1$, so $du = 2x\,dx$ and thus $x\,dx = \frac{1}{2}\,du$.

$$\begin{aligned}
\int \frac{x^4+x^2+1}{x^3+x}\,dx &= \int \left(x + \frac{1}{x^3+x}\right)dx = \int x\,dx + \int \frac{1}{x^3+x}\,dx \\
&= \frac{x^2}{2} + \int \left(\frac{-x}{x^2+1} + \frac{1}{x}\right)dx = \frac{x^2}{2} - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x}\,dx \\
&= \frac{x^2}{2} - \int \frac{1}{u}\cdot\frac{1}{2}\,du + \ln(x) = \frac{x^2}{2} - \frac{1}{2}\ln(u) + \ln(x) + C \\
&= \frac{x^2}{2} - \frac{1}{2}\ln\left(x^2+1\right) + \ln(x) + C \quad \square
\end{aligned}$$

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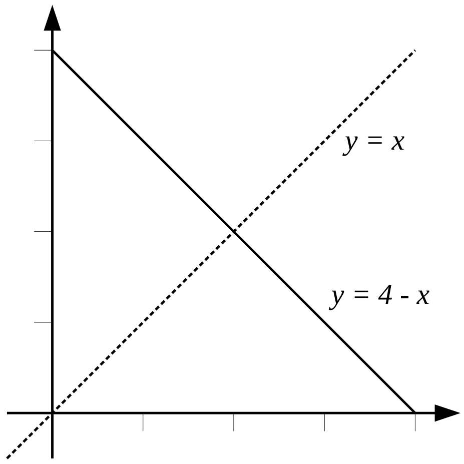
* That is, fortunately for you: this problem was very carefully cooked . . .

b. Note that the triangular plate is symmetric about the line $y = x$. Since the centroid must be on any line of symmetry for a plate of constant thickness and density, the centroid $(\bar{x}, \bar{y})$ of the given plate must have $\bar{x} = \bar{y}$. This means we only need to compute $\bar{x}$.



We will need to compute the mass of the plate, which boils down to computing its area since the plate is of uniform density and thickness and we are allowed to assume that units have been chosen to make mass equal to area. As noted in the diagram above, the line joining $(0,4)$ and $(4,0)$ has equation $y = 4 - x$, so the triangular plate can be thought of as the region under $y = 4 - x$ for $0 \le x \le 4$.

$$\begin{aligned} \text{mass} = \text{area} &= \int_0^4 (4-x)\,dx = \left(4x - \frac{x^2}{2}\right)\Big|_0^4 = \left(4 \cdot 4 - \frac{4^2}{2}\right) - \left(4 \cdot 0 - \frac{0^2}{2}\right) \\ &= \left(16 - \frac{16}{2}\right) - 0 = 16 - 8 = 8 \end{aligned}$$

We next need to compute the (first) moment of the plate with respect to x, which weighs area according to its x-value.

$$\begin{aligned} \text{moment} &= \int_0^4 (4-x)x\,dx = \int_0^4 \left(4x - x^2\right)\,dx = \left(4 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^4 \\ &= \left(4 \cdot \frac{4^2}{2} - \frac{4^3}{3}\right) - \left(4 \cdot \frac{0^2}{2} - \frac{0^3}{3}\right) = \left(32 - \frac{64}{3}\right) - 0 = \frac{96}{3} - \frac{64}{3} = \frac{32}{3} \end{aligned}$$

By definition, it now follows that $\bar{x} = \dfrac{\text{moment}}{\text{mass}} = \dfrac{32/3}{8} = \dfrac{4}{3}$. As noted at the start, $\bar{y} = \bar{x}$ by symmetry, so the centroid of the plate is at $(\bar{x}, \bar{y}) = \left(\dfrac{4}{3}, \dfrac{4}{3}\right)$. ■

[Total = 30]