Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2019

Assignment #6
Powerfully Serious Stuff
Due on Friday, 5 April.

1. Find a power series that is equal to $f(x) = \frac{1}{1+x^2}$ when it converges and determine its radius and interval of convergence. *[3]*

Hint: Think of $\frac{1}{1+x^2}$ as the sum of a geometric series.

SOLUTION. Recall that for a geometric series with first term a and common ratio r we have $\frac{a}{1-r} = a + ar + ar^2 + ar^3 + \cdots$, so long as the series converges, which happens exactly when $|r| < 1$. Writing $\frac{1}{1+x^2}$ as $\frac{1}{1-(-x^2)}$, we see that is is the sum of a geometric series with first term $a = 1$ and common ratio $r = -x^2$. Thus

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

which converges exactly when $|r| = \left|-x^2\right| = x^2 < 1$, *i.e.* exactly when $-1 < x < 1$. It follows that this power series has radius of convergence $R = 1$ and interval of convergence $(-1, 1)$. $\square$

2. Use the power series you obtained in **1** to find a power series that is equal to $\arctan(x)$ when it converges and determine its radius and interval of convergence. *[3]*

Hint: Integrate term-by-term.

SOLUTION. Recall that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$. Since $\arctan(0) = 0$, it follows that $\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt$. Using our solution to **1** and applying the hint, we get:

$$\begin{aligned}
\arctan(x) &= \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \left[1 - t^2 + t^4 - t^6 + \cdots\right]\,dt \\
&= \int_0^x 1\,dt - \int_0^x t^2\,dt + \int_0^x t^4\,dt - \int_0^x t^6\,dt + \cdots \\
&= t\Big|_0^x - \frac{t^3}{3}\Big|_0^x + \frac{t^5}{5}\Big|_0^x - \frac{t^7}{7}\Big|_0^x + \cdots \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}
\end{aligned}$$

Sadly, this is not a geometric series, so we need to a little work to determine the radius and interval of convergence. As usual, we will use the Ratio Test to determine the radius of convergence:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2(n+1)+1}}{2(n+1)+1}}{\frac{(-1)^n x^{2n+1}}{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{2n+3}\cdot\frac{2n+1}{(-1)^n x^{2n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{-x^2(2n+1)}{2n+3}\right| = \lim_{n\to\infty} x^2\cdot\frac{2n+1}{2n+3} = x^2\cdot\lim_{n\to\infty}\frac{2n+1}{2n+3}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} \\
&= x^2\cdot\lim_{n\to\infty}\frac{2+\frac{1}{n}}{2+\frac{3}{n}} = x^2\cdot\frac{2+0}{2+0} = x^2\cdot 1 = x^2
\end{aligned}$$

It follows by the Ratio Test that the series converges absolutely when $x^2 < 1$, *i.e.* when $|x| < 1$, and diverges when $x^2 > 1$, *i.e.* when $|x| > 1$, so its radius of convergence is $R = 1$. To find the interval of convergence, we need to check what happens when $x^2 = 1$, *i.e.* when $x = -1$ and when $x = 1$.

When $x = -1$, our series is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ because $(-1)^{2n} = 1$ for all $n \geq 0$. This is an alternating series – $(-1)^{n+1}$ alternates and $\dfrac{1}{2n+1}$ is positive – with decreasing absolute values – $\dfrac{1}{2(n+1)+1} < \dfrac{1}{2n+1}$ because $2(n+1)+1 = 2n+3 > 2n+1$ – and whose terms have a limit of 0 – $\displaystyle\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{2n+1}\right| = \lim_{n\to\infty}\frac{1}{2n+1} = 0$ since $2n+1 \to \infty$ as $n \to \infty$. It follows that it converges by the Alternating Series Test. Since the corresponding series of positive terms, $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2n+1}$, diverges by the Generalized p-Test because $p = 1 - 0 = 1 \not> 1$, $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ converges conditionally.

When $x = 1$, our series is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n 1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$. This is just the negative of the series for $x = -1$ (since $(-1)^{n+1} = -(-1)^n$), so it also converges conditionally.

It follows from the above that the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ is $[-1, 1]$. □

3. Use the power series you obtained in **2** to find a series summing to π. How many terms of this series would you need to ensure that the partial sum is within 0.001 of π? *[4]*

Hint: Hmm – what is $\arctan(1)$ equal to? For the second part, read up on the finer details of alternating series.

SOLUTION. Following the hint, and using our power series for $\arctan(x)$:

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Multiplying both sides by 4 gives us:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

To estimate how many terms of the series are needed to ensure that the partial sum up to that point is within $0.001 = \frac{1}{1000}$ of π we turn to the discussion at the end of §11.4 on how to do the like with any alternating series. To summarize: If $\sum_{n=0}^{\infty} a+n$ is an alternating series that passes the Alternating Series Test and hence converges to some number A, then A is always between any two consecutive partial sums $\sum_{n=0}^{N} a_n$ and $\sum_{n=0}^{N+1} a_n$ of the series. It follows that any partial sum $\sum_{n=0}^{N} a_n$ is within the absolute value of the next term, *i.e.* within $|a_{N+1}|$, of the sum A of the entire series.

We now apply this observation to our series summing to π. The Nth partial sum $\sum_{n=0}^{N} (-1)^n \frac{4}{2n+1}$ is guaranteed to be within $\left|(-1)^{N+1} \frac{4}{2(N+1)+1}\right| = \frac{4}{2N+3}$ of π. We therefore need to find the (first, just to be efficient) N such that $\frac{4}{2N+3} \leq \frac{1}{1000}$. This will happen if $2N+3 \geq 4000$, which will happen if $N \geq \frac{4000-3}{2} = \frac{3997}{2} = 1998.5$. The first integer meeting this condition is $N = 1999$. We must therefore sum at least the first 1999 terms of the series to guarantee that the partial sum will be within 0.001 of π. ■

NOTE: The series you (hopefully!) obtained in **2** is often called *Gregory's series* after James Gregory, who rediscovered it in 1668. It had been previously discovered by Madhava of Sangamagrama (*c.* 1340 – *c.* 1425), a mathematician and astronomer from Kerala in southern India. He also obtained the series formula for π in **3**. Both the power series and the series formula for π were also rediscovered by Gottfried Leibniz in the 1670s.