Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2019

Solutions to Assignment #3
Series, inverse squares, and trig

Your task on this assignment will be to show that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

1. Verify the following trigonometric identity. (So long as x is not an integer multiple of π anyway! :-) *[2]*

$$\frac{1}{\sin^2(x)} = \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{\sin^2\left(\frac{x+\pi}{2}\right)}\right]$$

Hint: Use common trig identities and the fact that for any t, $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$.

SOLUTION. Recall that $\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1 = 1 - 2\sin^2(t)$, from which it follows that $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$ and $\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$. we will use the last two identities and the one given in the hint with $t = \frac{x}{2}$.

$$\begin{aligned}
\frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{\sin^2\left(\frac{x+\pi}{2}\right)}\right] &= \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{\cos^2\left(\frac{x}{2}\right)}\right] \\
&= \frac{1}{4}\left[\frac{1}{\frac{1}{2}\left(1 - \cos\left(2\frac{x}{2}\right)\right)} + \frac{1}{\frac{1}{2}\left(1 + \cos\left(2\frac{x}{2}\right)\right)}\right] \\
&= \frac{1}{4}\left[\frac{1}{\frac{1}{2}(1 - \cos(x))} + \frac{1}{\frac{1}{2}(1 + \cos(x))}\right] \\
&= \frac{1}{4} \cdot \frac{\frac{1}{2}(1 + \cos(x)) + \frac{1}{2}(1 - \cos(x))}{\frac{1}{2}(1 - \cos(x)) \cdot \frac{1}{2}(1 + \cos(x))} \\
&= \frac{1}{4} \cdot \frac{1}{\frac{1}{4}(1 - \cos^2(x))} = \frac{1}{1 - \cos^2(x)} = \frac{1}{\sin^2(x)} \quad \square
\end{aligned}$$

2. Verify the following trigonometric summation formula for $m \geq 1$. *[2]*

$$1 = \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)}$$

Hint: Apply the identity from question **1** repeatedly, starting from $1 = \dfrac{1}{\sin^2\left(\frac{\pi}{2}\right)}$. You may find the fact that $\sin(t) = \sin(\pi - t)$ comes in handy.

Solution. Let's follow the hint to see what happens. Note that $\sin\left(\frac{\pi}{2}\right) = 1$.

$$\begin{aligned}
1 &= \frac{1}{1^2} = \frac{1}{\sin^2\left(\frac{\pi}{2}\right)} = \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\frac{\pi}{2}}{2}\right)} + \frac{1}{\sin^2\left(\frac{\frac{\pi}{2}+\pi}{2}\right)}\right] = \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{4}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{4}\right)}\right] \\
&= \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{4}\right)} + \frac{1}{\sin^2\left(\pi - \frac{3\pi}{4}\right)}\right] = \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{4}\right)} + \frac{1}{\sin^2\left(\frac{\pi}{4}\right)}\right] = \frac{1}{4}\cdot\frac{2}{\sin^2\left(\frac{\pi}{4}\right)} \\
&= \frac{2}{4}\cdot\frac{1}{\sin^2\left(\frac{\pi}{4}\right)} = \frac{2}{4^1}\sum_{k=0}^{0}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{1+1}}\right)} = \frac{2}{4^1}\sum_{k=0}^{2^{1-1}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{1+1}}\right)}
\end{aligned}$$

This gives us the instance $m = 1$ of the formula. What happens if we apply the formula from **1** again?

$$\begin{aligned}
1 &= \frac{2}{4^1}\sum_{k=0}^{2^{1-1}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{1+1}}\right)} = \frac{2}{4}\cdot\frac{1}{\sin^2\left(\frac{\pi}{4}\right)} = \frac{2}{4}\cdot\frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\frac{\pi}{4}}{2}\right)} + \frac{1}{\sin^2\left(\frac{\frac{\pi}{4}+\pi}{2}\right)}\right] \\
&= \frac{2}{4}\cdot\frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{5\pi}{8}\right)}\right] = \frac{2}{4}\cdot\frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\pi - \frac{5\pi}{8}\right)}\right] \\
&= \frac{2}{4}\cdot\frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{8}\right)}\right] = \frac{2}{4^2}\sum_{k=0}^{1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{2+1}}\right)} = \frac{2}{4^2}\sum_{k=0}^{2^{2-1}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{2+1}}\right)}
\end{aligned}$$

Note that $2^{2-1} - 1 = 1$, so the sum includes two terms, one for $k = 0$ and one for $k = 1$; this time we got the instance $m = 2$ of the formula we want to obtain. Let's try it one more time, skipping a step or two:

$$\begin{aligned}
1 &= \frac{2}{4^2}\sum_{k=0}^{2^{2-1}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{2+1}}\right)} = \frac{2}{4^2}\left[\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{8}\right)}\right] \\
&= \frac{2}{4^2}\left[\frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{16}\right)} + \frac{1}{\sin^2\left(\pi - \frac{9\pi}{16}\right)}\right] + \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{3\pi}{16}\right)} + \frac{1}{\sin^2\left(\pi - \frac{11\pi}{8}\right)}\right]\right] \\
&= \frac{2}{4^3}\left[\frac{1}{\sin^2\left(\frac{\pi}{16}\right)} + \frac{1}{\sin^2\left(\frac{7\pi}{16}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{16}\right)} + \frac{1}{\sin^2\left(\frac{5\pi}{16}\right)}\right] \\
&= \frac{2}{4^3}\left[\frac{1}{\sin^2\left(\frac{\pi}{16}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{16}\right)} + \frac{1}{\sin^2\left(\frac{5\pi}{16}\right)} + \frac{1}{\sin^2\left(\frac{7\pi}{16}\right)}\right] = \frac{2}{4^3}\sum_{k=0}^{2^{3-1}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{3+1}}\right)}
\end{aligned}$$

Note that $2^{3-1} - 1 = 3$, so the sum includes four terms, one each for $k = 0$, 1, 2, and 3; this time we got the instance $m = 3$ of the formula.

Examining these calculations reveals some patterns that continue to work in the general case. Each application of the identity from **1** brings out another factor of $\frac{1}{4}$ and doubles the number of terms in the sum. In half of the terms one has to use the identity $\sin(t) = \sin(\pi - t)$ to put them in a form that can be combined with the other half of the terms into the desired sum. Let's see what we can do with these insights in the general case:

$$
\begin{aligned}
1 &= \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)} = \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{m+1}\cdot 2}\right)} + \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{m+1}\cdot 2} + \frac{\pi}{2}\right)}\right] \\
&= \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{(m+1)+1}}\right)} + \frac{1}{\sin^2\left(\pi - \left(\frac{(2k+1)\pi}{2^{(m+1)+1}} + \frac{\pi}{2}\right)\right)}\right] \\
&= \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{(m+1)+1}}\right)} + \frac{1}{\sin^2\left(\frac{\pi}{2} - \frac{(2k+1)\pi}{2^{(m+1)+1}}\right)}\right] \\
&= \frac{2}{4^{m+1}} \sum_{k=0}^{2^{m-1}-1} \left[\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{(m+1)+1}}\right)} + \frac{1}{\sin^2\left(\frac{(2^{m+1}-2k-1)\pi}{2^{(m+1)+1}}\right)}\right] \\
&= \frac{2}{4^{m+1}} \left[\left(\sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{(m+1)+1}}\right)}\right) + \left(\sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2^{m+1}-2k-1)\pi}{2^{(m+1)+1}}\right)}\right)\right]
\end{aligned}
$$

Note that the first sum on the last line is the first half of the desired sum. We will rename the variable k in the second sum to ℓ to make it more convenient to compare it to the second half of the desired sum without getting confused. Our objective now is to check that

$$
\sum_{\ell=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2^{m+1}-2\ell-1)\pi}{2^{(m+1)+1}}\right)} = \sum_{k=2^{m-1}}^{2^{(m+1)-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{(m+1)+1}}\right)}.
$$

Observe that the denominators of the inputs to $\sin(x)$ are the same in both sums, namely $2^{(m+1)+1}$. To show that the sums are the same, therefore, it suffices to show that the numerators run through the same values in each sum, which they do, albeit in reverse order. To see this, observe that $2^{m+1} - 2\ell - 1 = 2\left(2^m - \ell - 1\right) + 1$. If we set $k = 2^m - \ell - 1$, then as ℓ runs up from 0 to $2^{m-1} - 1$, k will run from $2^m - 0 - 1 = 2^{(m+1)-1} - 1$ down to $2^m - \left(2^{m-1} - 1\right) - 1 = 2^m - 2^{m-1} + 1 - 1 = 2^{m-1}$, as desired. (Note that $2^m = 2 \cdot 2m - 1$, so $2^m - 2^{m-1} - 2^{m-1}$.) It now follows that:

$$\begin{aligned}
1 &= \frac{2}{4^{m+1}}\left[\left(\sum_{k=0}^{2^{m-1}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{(m+1)+1}}\right)}\right)+\left(\sum_{\ell=0}^{2^{m-1}-1}\frac{1}{\sin^2\left(\frac{(2^{m+1}-2\ell-1)\pi}{2^{(m+1)+1}}\right)}\right)\right]\\
&= \frac{2}{4^{m+1}}\left[\left(\sum_{k=0}^{2^{m-1}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{(m+1)+1}}\right)}\right)+\left(\sum_{k=2^m}^{2^{(m+1)-1}-1}\frac{1}{\sin^2\left(\frac{(2^{m+1}-2k-1)\pi}{2^{(m+1)+1}}\right)}\right)\right]\\
&= \frac{2}{4^{m+1}}\sum_{k=0}^{2^{(m+1)-1}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{(m+1)+1}}\right)} \qquad \square
\end{aligned}$$

3. Verify the following limit formula, where $k \geq 0$ is fixed. *[2]*

$$\lim_{m\to\infty} 2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right) = \frac{(2k+1)\pi}{2}$$

Hint: This is really just (a version of) $\lim_{t\to 0}\frac{\sin(t)}{t} = 0 \ldots$ *[Oops! That was a typo the limit equals* 1*, not* 0*.]*

SOLUTION. A little rearranging, a little algebra, and one use of l'Hôpital's Rule:

$$\begin{aligned}
\lim_{m\to\infty} 2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right) &= \lim_{m\to\infty}\frac{\sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)}{\frac{1}{2^m}} = \lim_{m\to\infty}\frac{\sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)}{\frac{1}{2^m}}\cdot\frac{\frac{(2k+1)\pi}{2}}{\frac{(2k+1)\pi}{2}}\\
&= \lim_{m\to\infty}\frac{\frac{(2k+1)\pi}{2}\sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)}{\frac{(2k+1)\pi}{2^{m+1}}} \qquad \text{Now let } t = \frac{(2k+1)\pi}{2^{m+1}}, \text{ then } t \to 0 \text{ as } m\to\infty.\\
&= \lim_{t\to 0}\frac{\frac{(2k+1)\pi}{2}\sin(t)}{t} = \frac{(2k+1)\pi}{2}\lim_{t\to 0}\frac{\sin(t)}{t}\begin{matrix}\to 0\\ \to 0\end{matrix}
\end{aligned}$$

Since both numerator and denominator go to 0, we can use l'Hôpital's Rule.

$$\begin{aligned}
&= \frac{(2k+1)\pi}{2}\lim_{t\to 0}\frac{\frac{d}{dt}\sin(t)}{\frac{d}{dt}t} = \frac{(2k+1)\pi}{2}\lim_{t\to 0}\frac{\cos(t)}{1}\\
&= \frac{(2k+1)\pi}{2}\cos(0) = \frac{(2k+1)\pi}{2}\cdot 1 = \frac{(2k+1)\pi}{2} \qquad \square
\end{aligned}$$

4. Take the limit as $m\to\infty$ of the identity in **2**, and use **3** to show the following. *[2]*

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

NOTE: Here you will need to interchange a limit with a sum, which you may do without having to justify it. (That's the one thing in this argument that is not really first-year-calculus-level material.)

SOLUTION. We'll follow the hint and see where it goes. Note that $2^{m-1} - 1 \to \infty$ as $m \to \infty$, so $\displaystyle\lim_{m\to\infty} \sum_{k=0}^{2^{m-1}-1} \cdots$ ought to be equal to $\displaystyle\sum_{k=0}^{\infty} \lim_{m\to\infty} \cdots$.

$$\begin{aligned}
1 &= \lim_{m\to\infty} 1 = \lim_{m\to\infty} \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)} = \lim_{m\to\infty} \sum_{k=0}^{2^{m-1}-1} \frac{2}{4^m \sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)} \\
&= \sum_{k=0}^{\infty} \lim_{m\to\infty} \frac{2}{\left(2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)\right)^2} = \sum_{k=0}^{\infty} \frac{2}{\left(\lim_{m\to\infty} 2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)\right)^2} = \sum_{k=0}^{\infty} \frac{2}{\left(\frac{(2k+1)\pi}{2}\right)^2} \\
&= \sum_{k=0}^{\infty} \frac{8}{(2k+1)^2\pi^2} = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}
\end{aligned}$$

Comparing the first and last in this chain of equalities and solving for the sum, we get that $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$, as desired. $\square$

5. Use **4** and some algebra to check that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

is true. *[2]*

Hint: Split up $\sum_{n=1}^{\infty} \frac{1}{n^2}$ into the sums of the terms for even and odd n respectively and try to rewrite the sum of the terms for even n.

SOLUTION. We'll follow the hint and see what happens.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{m=1}^{\infty} \frac{1}{(2m)^2} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \sum_{m=1}^{\infty} \frac{1}{4m^2} + \frac{\pi^2}{8} = \frac{1}{4} \sum_{m=1}^{\infty} \frac{1}{m^2} + \frac{\pi^2}{8}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{m=1}^{\infty} \frac{1}{m^2}$, it follows that $\displaystyle\left(1 - \frac{1}{4}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$, and hence that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}$, as desired. $\blacksquare$