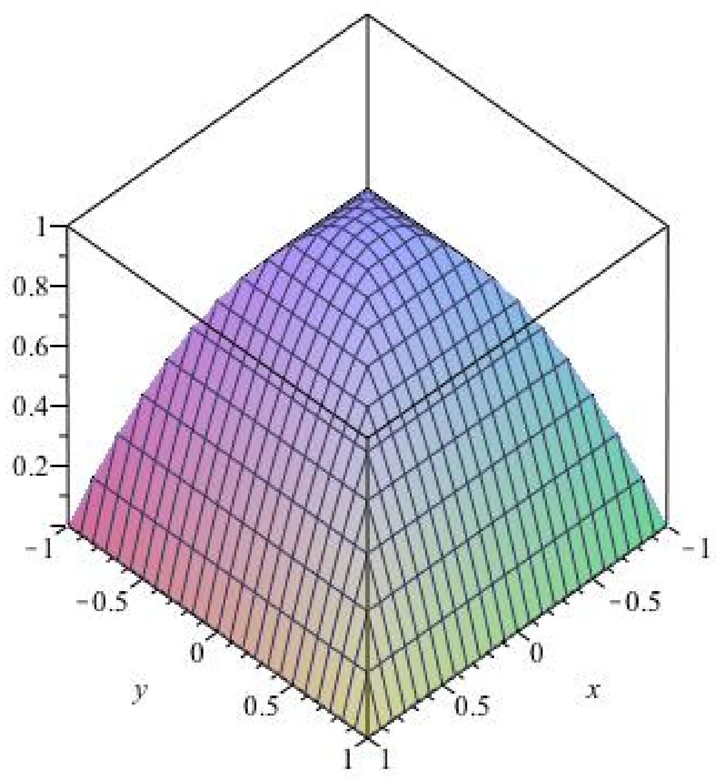
Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2019

Solution to Assignment #2
Plumbing Volume

1. Two cylinders, each of radius 1, intersect such that their axes of symmetry meet at right angles. What is the volume of the region common to both cylinders? *[10]*

Solution. We first set this up in three dimensions with the usual Cartesian coordinates. The x-, y-, and z-axes meet at right angles and we will use the first two as the axes of symmetry of the two cylinders. The cylinder of radius 1 whose axis of symmetery is the x-axis is given by the equation $y^2 + z^2 = 1$ in three dimensions (it is swept out by the circle with that equation in the yz-plane as that circle is moved parallel to the x-axis); similarly, the cylinder of radius 1 whose axis of symmetry is the y-axis is given by the equation $x^2 + z^2 = 1$ – (it is swept out by the circle with that equation in the xz-plane as that circle is moved parallel to the y-axis). The (interiors of the) two cylinders intersect in a region containing the origin. Here is picture of the upper half of the region in question, drawn by `Maple`.

```
> plot3d(min(1-x^2,1-y^2),x=-1..1,y=-1..1,axes=boxed)
```



(The `min` function was used to show just the region cut off by both cylinders.)

Looking at the plot, it is not hard to guess that the cross-sections of the region parallel to the xy-plane, *i.e.* those with a fixed value of z, are rectangles. In fact, because the definition of the region is symmetric in terms of x and y, these cross-sections are squares. If we solve the equations of the two cylinders for x and y, respectively, using the same value of z we get that $x = \pm\sqrt{1-z^2}$ and $y = \pm\sqrt{1-z^2}$. This means that over the interior of the cross-section we have both $-\sqrt{1-z^2} \leq x \leq +\sqrt{1-z^2}$ and $-\sqrt{1-z^2} \leq y \leq +\sqrt{1-z^2}$; it follows that the length of the sides parallel to each axis of the cross-section are each of length $\sqrt{1-z^2} - \left(-\sqrt{1-z^2}\right) = 2\sqrt{1-z^2}$. Thus the area of the cross-section of the region for a fixed value of z is $A(z) = 2\sqrt{1-z^2} \cdot 2\sqrt{1-z^2} = 4\left(1-z^2\right)$.

Notice that the possible values of z over the region are $-1 \leq z \leq 1$; these are the minimum to the maximum values of z for a cylinder of radius 1 centered on the x- or y-axis.

Having waded through the set-up, the integral for the volume of the solid is pretty easy to compute. As usual, we integrate the areas of cross-sections to get the volume:

$$
\begin{aligned}
V &= \int_{-1}^{1} A(z)\,dz = \int_{-1}^{1} 4\left(1-z^2\right)\,dz = 4\int_{-1}^{1} \left(1-z^2\right) = 4\left(z - \frac{z^3}{3}\right)\Bigg|_{-1}^{1} \\
&= 4\left(1 - \frac{1^3}{3}\right) - 4\left((-1) - \frac{(-1)^3}{3}\right) = 4 \cdot \frac{2}{3} - 4 \cdot \left(-\frac{2}{3}\right) = \frac{8}{3} + \frac{8}{3} = \frac{16}{3} \quad \blacksquare
\end{aligned}
$$