Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2019

Solutions to Assignment #1
A little bit about the Riemann integral
Due on Friday, 18 January.

Recall from MATH 1110H [especially Assignment #9], class, or the textbook, that the definite integral $\int_a^b f(x)\,dx$ essentially gives the signed or weighted area of the region between $y = f(x)$ and the x-axis, where area above the x-axis is added and area below the x-axis is subtracted. The definite integral is usually defined in terms of limits of Riemann sums, but the full general definition is pretty cumbersome to work with. This assignment is meant to give you a little bit of practice with it and give an inkling as to why simplifications like the Right-Hand Rule are not quite enough to justify all the properties of definite integrals.

First, here is the aforementioned Right-Hand Rule, which will, in principle, properly compute $\int_a^b f(x)\,dx$ for most commonly encountered functions.

RIGHT-HAND RULE. Suppose $f(x)$ is defined for all x in $[a, b]$ and is continuous at all but finitely many points of $[a, b]$. Then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i \cdot \frac{b-a}{n}\right)\right]$$

The idea here is that we divide up the interval $[a, b]$ into n subintervals of equal width $\frac{b-a}{n}$, so the ith subinterval, going from left to right and where $1 \le i \le n$, will be $\left[(i-1)\cdot\frac{b-a}{n}, i\cdot\frac{b-a}{n}\right]$. Each subinterval serves as the base of a rectangle of height $f\left(a + i\cdot\frac{b-a}{n}\right)$, which must then have area $\frac{b-a}{n} f\left(a + i\cdot\frac{b-a}{n}\right)$. The sum of the areas of these rectangles, the *nth Right-Hand Rule sum for* $\int_a^b f(x)\,dx$, approximates the area computed by $\int_a^b f(x)\,dx$. (It's called the Right-Hand Rule because it uses the right-hand endpoint of each subinterval to evaluate $f(x)$ at to determine the height of the rectangle which has that subinterval as a base.) As we increase n and so shrink the width of the rectangles we get better and better approximations to the definite integral.

1. Use the Right-Hand Rule to show that $\displaystyle\int_0^a e^x\,dx = e^a$. *[3]*

Hint. You'll need to do some algebra and may want to look up geometric series (Example 11.2.1 in the textbook) and their summation formulas if you don't remember them.

SOLUTION. We plug the interval $[0, a]$ and the function $f(x) = e^x$ into the Right-Hand Rule formula and chug away:

$$\begin{aligned}\int_0^a e^x\,dx &= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{a-0}{n} \cdot e^{0+i\cdot\frac{a-0}{n}}\right] = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{a}{n} \cdot e^{ia/n}\right] \\ &= \lim_{n\to\infty}\left[\frac{a}{n}\sum_{i=1}^{n} e^{ia/n}\right] = \lim_{n\to\infty}\left[\frac{a}{n}\sum_{i=1}^{n} \left(e^{a/n}\right)^i\right]\end{aligned}$$

Per the hint, observe that

$$\sum_{i=1}^{n} \left(e^{a/n}\right)^i = e^{a/n} + \left(e^{a/n}\right)^2 + \left(e^{a/n}\right)^3 + \cdots + \left(e^{a/n}\right)^n$$

is a geometric series – *i.e.* a sum of the form $k + kx + kx^2 + \cdots kx^m$ – with first term $k = e^{a/n}$, common ratio $x = e^{a/n}$, and final power $m = n-1$. As noted in Example 11.2.1 of the text, this sum is equal to :

$$k\frac{x^{m+1}-1}{x-1} = e^{a/n}\frac{\left(e^{a/n}\right)^{n-1+1}-1}{e^{a/n}-1} = e^{a/n}\frac{\left(e^{a/n}\right)^{n}-1}{e^{a/n}-1} = e^{a/n}\frac{e^a-1}{e^{a/n}-1}$$

We replace the sum by this formula in the limit, so:

$$\int_0^a e^x\,dx = \lim_{n\to\infty}\left[\frac{a}{n}\sum_{i=1}^{n}\left(e^{a/n}\right)^i\right] = \lim_{n\to\infty}\left[\frac{a}{n}e^{a/n}\frac{e^a-1}{e^{a/n}-1}\right]$$

To make it a easier to evaluate we substitute $h = \frac{a}{n}$ in the limit, which also changes it from a limit as $n \to \infty$ to a limit as $h \to 0$. It follows, with a bit of help from the rules for manipulating limits, including l'Hôpital's Rule, that:

$$\begin{aligned}
\int_0^a e^x\,dx &= \lim_{n\to\infty}\left[\frac{a}{n}e^{a/n}\frac{e^a-1}{e^{a/n}-1}\right] = \lim_{h\to 0}\left[he^h\frac{e^a-1}{e^h-1}\right] = (e^a-1)\lim_{h\to 0}\left[e^h\frac{h}{e^h-1}\right] \\
&= (e^a-1)\left(\lim_{h\to 0}e^h\right)\left(\lim_{h\to 0}\frac{h}{e^h-1}\right) = (e^a-1)\left(e^0\right)\left(\lim_{h\to 0}\frac{h}{e^h-1}\begin{matrix}\to 0\\ \to 0\end{matrix}\right) \\
&= (e^a-1)(1)\left(\lim_{h\to 0}\frac{\frac{d}{dh}h}{\frac{d}{dh}\left(e^h-1\right)}\right) = (e^a-1)\left(\lim_{h\to 0}\frac{1}{e^h}\right) = (e^a-1)\left(\frac{1}{e^0}\right) \\
&= (e^a-1)\left(\frac{1}{1}\right) = e^a - 1 \qquad \blacksquare
\end{aligned}$$

2. Suppose $f(x)$ is a function which is defined and continuous – and hence ought to be integrable – on all of $\mathbb{R}$. Explain why we would have a problem justifying

$$\int_{-1}^{0} f(x)\,dx + \int_{0}^{\sqrt{2}} f(x)\,dx = \int_{-1}^{\sqrt{2}} f(x)\,dx$$

if we used the Right-Hand Rule (or any rule that relies on subdividing $[a,b]$ into equal subintervals) as the actual definition of $\int_a^b f(x)\,dx$. *[3]*

Hint. It matters here that $\sqrt{2}$ is irrational.

Solution. The problem here is that the Right-Hand Rule relies on partitions in which each subinterval is of equal width. To make it easy to combine the Right-Hand Rule limits for $\int_{-1}^{0} f(x)\,dx$ and $\int_{0}^{\sqrt{2}} f(x)\,dx$ into a Right-Hand Rule limit for $\int_{-1}^{\sqrt{2}} f(x)\,dx$, you have to be able to combine suitable Right-Hand Rule sums for the first two integrals into a Right-Hand Rule sum for the third. However, if you partition $[-1,0]$ into n equal pieces, each will have width $\frac{1}{n}$, and if you partition $\left[0,\sqrt{2}\right]$ into k equal pieces, each will have width $\frac{\sqrt{2}}{k}$. There is no way $\frac{1}{n}=\frac{\sqrt{2}}{k}$, no matter what integers n and k happen to be: if it is otherwise, then we would have $\sqrt{2}=\frac{k}{n}$, which would mean that $\sqrt{2}$ was rational, and it isn't.

There is a similar problem splitting up a Right-Hand Rule sum for $\int_{-1}^{\sqrt{2}} f(x)\,dx$ into Right-Hand Rule sums for $\int_{-1}^{0} f(x)\,dx$ and $\int_{0}^{\sqrt{2}} f(x)\,dx$. No matter how many pieces of equal width you divide $\left[-1,\sqrt{2}\right]$ into, all of the partition points (except -1) will be irrational, which means that 0 can't be partition point ...

Similar problems occur for any would-be definition of the definite integral that relies on partitions of equal width. ■

The full definition of the Riemann integral uses the same basic idea of approximating the area under a curve by rectangles, but it allows for these rectangles to have differing widths and heights that are computed by evaluating the function at arbitrary points in the bases of the rectangles. This, sadly, means that we need some preliminary definitions and notation.

PARTITIONS. If $[a,b]$ is a (closed finite) interval, then a *partition* of $[a,b]$ is a set of subintervals $[x_0,x_1]$, $[x_1,x_2]$, ... , $[x_{n-1},x_n]$, of $[a,b]$ where $n\geq 1$ and $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$. (Note that different partitions may divide up $[a,b]$ into different numbers of subintervals n.) For the sake of brevity, we will often denote such a partition by $\{x_i\}$.

If $\{x_i\}$ is a partition of $[a,b]$, its *norm* $\|\{x_i\}\|$ is the maximum width of a subinterval of the partition, *i.e.* $\|\{x_i\}\|=\max\{\, x_i-x_{i-1} \mid 1\leq i\leq n \,\}$.

A *tagged partition* of $[a,b]$ is a partition $\{x_i\}$ of $[a,b]$ together with some choice of points x_i^* for each i with $1\leq i\leq n$, where the only restriction is that $x_{i-1}\leq x_i^*\leq x_i$ for each i. For the sake of brevity, we will often denote a tagged partition by $\{x_i\}^*$

SUPREMA. If A is a set of real numbers with an *upper bound* u, *i.e.* $a\leq u$ for all $a\in A$, then A has a *least upper bound* or *supremum*, often denoted by $\sup(A)$, which is an upper bound of A and such that $\sup(A)\leq u$ for every other upper bound u of A. Note that if A is finite, then $\sup(A)=\max\{\, a \mid a\in A \,\}\in A$, but this doesn't have to be true if A is infinite. For example, $\sup\left(\,(0,1)\,\right)=1\notin(0,1)$.

THE RIEMANN INTEGRAL. If $f(x)$ is a function defined on $[a,b]$ and $\{x_i\}^*$ is a tagged partition of $[a,b]$, the corresponding *Riemann sum* is $R\left(f,\{x_i\}^*\right)=\displaystyle\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)f\left(x_i^*\right)$. This is just the sum of the areas of n rectangles, where the base of the ith rectangle is the subinterval $[x_{i-1},x_i]$ and its height is $f\left(x_i^*\right)$. Note that Right-Hand Rule sums are Riemann sums for certain very special tagged partitions.

If $f(x)$ is a function defined on $[a,b]$, then its *Riemann integral* on $[a,b]$ is

$$\int_a^b f(x)\,dx=\lim_{\delta\to 0^+}\sup\left(\left\{\, R\left(f,\{x_i\}^*\right) \mid \|\{x_i\}\|<\delta \,\right\}\right)\,,$$

provided that this limit exists. If the limit does exist, then $f(x)$ is said to be *Riemann integrable* on $[a, b]$. Actually working with this definition is usually pretty hard because the definitional pile-up leading up to it means that the limit is very complicated. To make it worse, in practice you often have to go to the ε–δ definition of limits when working with it, too.

3. Suppose $f(x)$ is a function which is Riemann integrable on $\mathbb{R}$. Explain why the definition of the Riemann integral does justify

$$\int_{-1}^{0} f(x)\,dx + \int_{0}^{\sqrt{2}} f(x)\,dx = \int_{-1}^{\sqrt{2}} f(x)\,dx\,.$$

Your explanation should be informal, but as complete and precise as you can. *[3]*

Solution. The definitional pile-up that is the general Riemann integral is pretty complicated, but only one feature really matters here for us: the fact that the partitions used in defining the Riemann integral may have subintervals that are not of equal width. This fact means that a partition for $[-1, 0]$ and a partition for $\left[0, \sqrt{2}\right]$ can simply be stuck together to get a partition for $\left[-1, \sqrt{2}\right]$. Similarly, it is very easy to take a partition of $\left[-1, \sqrt{2}\right]$ and divide it into partitions for each of $[-1, 0]$ and $\left[0, \sqrt{2}\right]$; all you have to do (if 0 isn't already a partition point) is subdivide whatever subinterval 0 is in at 0. It follows, though proving this does require wading through the rest of the definitional pile-up at considerable length, that the limits of sums defining $\int_{-1}^{0} f(x)\,dx$ and $\int_{0}^{\sqrt{2}} f(x)\,dx$ equal the limit of sums defining $\int_{-1}^{\sqrt{2}} f(x)\,dx$. ■

0. Please give your name! No name, no mark … *[1]*

Comment. I really, *really,* hope you got this one right! :-)