Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2019
Solutions to the Final Examination
19:00-22:00 on Monday, 22 April, in the Gym.

Time: *3 hours.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **X**, **Y**, and **Z**, and, if you wish, part **W**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Any calculator; (all sides of) one aid sheet; one (1) brain (no neuron limit).

Part X. Do all four (4) of **1**–**4**.

1. Evaluate any *four* (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

a. $\displaystyle\int_{-1}^{1} \frac{1}{(x+2)^2}\,dx$ **b.** $\displaystyle\int z\cos(z)\,dz$ **c.** $\displaystyle\int \left(1-y^2\right)^{-1/2}\,dy$

d. $\displaystyle\int_{-\infty}^{0} 2ue^{u^2}\,du$ **e.** $\displaystyle\int \frac{1}{v^3+v}\,dv$ **f.** $\displaystyle\int_{0}^{\pi/2} \frac{\cos(w)}{\sin^2(w)+1}\,dw$

SOLUTIONS. **a.** *(Substitution)* We will use the substitution $u = x+2$, so $du = dx$ and change the limits as we go along: $\begin{matrix} x & -1 & 1 \\ u & 1 & 3 \end{matrix}$.

$$\int_{-1}^{1} \frac{1}{(x+2)^2}\,dx = \int_{1}^{3} \frac{1}{u^2}\,du = \int_{1}^{3} u^{-2}\,du = \left.\frac{u^{-1}}{-1}\right|_1^3 = \left.\frac{-1}{u}\right|_1^3 = \frac{-1}{3} - \frac{-1}{1} = \frac{2}{3} \quad \square$$

b. *(Integration by parts)* We will use integration by parts with $u = z$ and $v' = \cos(z)$, so $u' = 1$ and $v = \sin(z)$.

$$\begin{aligned}\int z\cos(z)\,dz &= z\sin(z) - \int \sin(z)\,dz = z\sin(z) - (-\cos(z)) + C \\ &= z\sin(z) + \cos(z) + C \quad \square\end{aligned}$$

c. *(Trigonometric substitution)* We will use the substitution $y = \sin(\theta)$, so $dy = \cos(\theta)\,d\theta$ and $\theta = \arcsin(y)$.

$$\begin{aligned}\int \left(1-y^2\right)^{-1/2}\,dy &= \int \frac{1}{\sqrt{1-y^2}}\,dy = \int \frac{1}{\sqrt{1-\sin^2(\theta)}}\cos(\theta)\,d\theta \\ &= \int \frac{1}{\sqrt{\cos^2(\theta)}}\cos(\theta)\,d\theta = \int \frac{\cos(\theta)}{\cos(\theta)}\,d\theta = \int 1\,d\theta \\ &= \theta + C = \arcsin(y) + C \quad \square\end{aligned}$$

d. *(Improper integral and substitution)* We will use the substitution $w = u^2$, so $dw = 2u\,du$, and substitute back after finding the antiderivative instead of changing the limits as we go along.

$$\int_{-\infty}^{0} 2ue^{u^2}\,du = \lim_{t\to-\infty} \int_{t}^{0} 2ue^{u^2}\,du = \lim_{t\to-\infty} \int_{u=t}^{u=0} e^w\,dw = \lim_{t\to-\infty} e^w\Big|_{u=t}^{u=0}$$
$$= \lim_{t\to-\infty} e^{u^2}\Big|_t^0 = \lim_{t\to-\infty} \left(e^{0^2} - e^{t^2}\right) \lim_{t\to-\infty} \left(1 - e^{t^2}\right) = -\infty$$

That is, this improper integral does not converge. □

e. *(Partial fractions)* Since the degree of the numerator, namely 0, is less than the degree of the denominator, namely 3, we can move directly to factoring the denominator. $v^3+v = v\left(v^2+1\right)$ and v^2+1 is an irreducible quadratic: since $v^2+1 \geq 1 > 0$ for all $v \in \mathbb{R}$, it has no roots and hence cannot be factored into linear factors with real coefficients.

The partial fraction decomposition is thus $\frac{1}{v^3+v} = \frac{1}{v\left(v^2+1\right)} = \frac{A}{v} + \frac{Bv+C}{v^2+1}$ for some unknown constants A, B, and C. Putting the right-hand side over a common denominator and then comparing numerators tells us that we must have $1 = Av^2+A+Bv^2+Cv = (A+B)v^2+Cv+A$, so $A+B=0$, $C=0$, and $A=1$, and thus $B=-A=-1$. We therefore have $\frac{1}{v^3+v} = \frac{1}{v} + \frac{-v}{v^2+1} = \frac{1}{v} - \frac{v}{v^2+1}$.

Now we can integrate. In one part we will use the substitution $w = v^2+1$, so $dw = 2v\,dv$ and hence $v\,dv = \frac{1}{2}\,dw$.

$$\int \frac{1}{v^3+v}\,dv = \int \left[\frac{1}{v} - \frac{v}{v^2+1}\right] dv = \int \frac{1}{v}\,dv - \int \frac{v}{v^2+1}\,dv$$
$$= \ln(v) - \int \frac{1}{w}\cdot\frac{1}{2}\,dw = \ln(v) - \frac{1}{2}\ln(w) + C$$
$$= \ln(v) - \frac{1}{2}\ln\left(v^2+1\right) + C \quad \square$$

f. *(Substitution)* We will use the substitution $u = \sin(w)$, so $du = \cos(w)\,dw$, and change the limits as we go along: $\begin{matrix} w & 0 & \pi/2 \\ u & 0 & 1 \end{matrix}$.

$$\int_0^{\pi/2} \frac{\cos(w)}{\sin^2(w)+1}\,dw = \int_0^1 \frac{1}{u^2+1}\,du = \arctan(u)\big|_0^1$$
$$= \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4} \quad \blacksquare$$

2. Determine whether the series converges in any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n+4^n}$ **b.** $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\ln(m+2)}$ **c.** $\displaystyle\sum_{\ell=2}^{\infty} \frac{e^\ell}{e^{2\ell}+1}$

d. $\displaystyle\sum_{k=3}^{\infty} \frac{(-1)^k 17^k}{k^k}$ **e.** $\displaystyle\sum_{j=4}^{\infty} \frac{j!}{(2j)!}$ **f.** $\displaystyle\sum_{i=5}^{\infty} \frac{(i+1)^3}{(i+2)^5}$

SOLUTIONS. **a.** *(Basic Comparison Test)* 2^n dominates the numerator and 4^n dominates the denominator, so the given series ought to converge since $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{4^n}$ does, which is the case because it is a geometric series with common ratio $r = \frac{2}{4} = \frac{1}{2}$, which has absolute value $\frac{1}{2} < 1$. Since making the denominator bigger makes a fraction smaller, we have $0 < \dfrac{2^n}{3^n+4^n} \leq \dfrac{2^n}{4^n}$ for all $n \geq 0$, so the given series converges by the Basic Comparison Test. □

b. *(Alternating Series Test)* First, since $\ln(m+2 > 0$ for all $m \geq 1$ while $(-1)^{m+1}$ alternates sign, the terms of the series, $\dfrac{(-1)^{m+1}}{\ln(m+2)}$, alternates.sign as well.

Second, since $\ln(x)$ is an icreasing function, it follows that

$$\left|\frac{(-1)^{(m+1)+1}}{\ln\left((m+1)+2\right)}\right| = \frac{1}{\ln\left(m+3\right)} < \frac{1}{\ln(m+2)} = \left|\frac{(-1)^{m+1}}{\ln(m+2)}\right|$$

for all $m \geq 1$.

Third, since $\ln(x) \to \infty$ as $x \to \infty$, it follows that

$$\lim_{m\to\infty} \left|\frac{(-1)^{m+1}}{\ln(m+2)}\right| = \lim_{m\to\infty} \frac{1}{\ln(m+2)} = 0\,.$$

Since the series satisfies all three hypotheses of the Alternating Series Test, it must converge. □

c. *(Limit Comparison Test)* e^ℓ dominates the numerator and $e^{2\ell}$ dominates the denominator, so the given series ought to converge since $\displaystyle\sum_{\ell=2}^{\infty} \frac{e^\ell}{e^{2\ell}} = \sum_{\ell=2}^{\infty} \frac{1}{e^\ell}$ converges, which it does because it is a geometric series with common ratio $r = \frac{1}{e}$, which has an absolute value of $\frac{1}{e} < 1$. Since

$$\lim_{\ell\to\infty} \frac{\frac{e^\ell}{e^{2\ell}+1}}{\frac{1}{e^\ell}} = \lim_{\ell\to\infty} \frac{e^\ell}{e^{2\ell}+1} \cdot \frac{e^\ell}{1} = \lim_{\ell\to\infty} \frac{e^{2\ell}}{e^{2\ell}+1} \cdot \frac{e^{-2\ell}}{e^{-2\ell}} = \lim_{\ell\to\infty} \frac{1}{1+e^{-2\ell}} = \frac{1}{1+0} = 1\,,$$

it follows that the given series must converge because $\displaystyle\sum_{\ell=2}^{\infty} \frac{1}{e^{\ell}}$ does. □

d. *(Root Test)* Since

$$\lim_{k\to\infty} \left| \frac{(-1)^k 17^k}{k^k} \right|^{1/k} = \lim_{k\to\infty} \left(\frac{17^k}{k^k} \right)^{1/k} = \lim_{k\to\infty} \left(\left[\frac{17}{k} \right]^k \right)^{1/k} = \lim_{k\to\infty} \frac{17}{k} = 0 < 1\,,$$

it follows by the Root Test that the given series converges. □

e. *(Ratio Test)* Since

$$\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} \left| \frac{\frac{(j+1)!}{(2(j+1))!}}{\frac{j!}{(2j)!}} \right| = \lim_{j\to\infty} \frac{(j+1)!}{(2j+2)!} \cdot \frac{(2j)!}{j!} = \lim_{j\to\infty} \frac{j+1}{(2j+2)(2j+1)}$$
$$= \lim_{j\to\infty} \frac{j+1}{2(j+1)(2j+1)} = \lim_{j\to\infty} \frac{1}{2(2j+1)} = \lim_{j\to\infty} \frac{1}{4j+2} = 0 < 1\,,$$

the given series converges by the Ratio Test. □

f. *(Generalized p-Test)* $\displaystyle\sum_{i=5}^{\infty} \frac{(i+1)^3}{(i+2)^5}$ converges by the Generalized p-Test because the numerator of each term is a polynomial of degree 3 and the denominator is a polynomial of degree 5, so $p = 5 - 3 = 2 > 1$. ■

3. Do any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. Use the Right-Hand Rule to compute $\displaystyle\int_0^2 (x+1)\,dx$.

b. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or diverges.

c. Find the area of the region between $y = \sqrt{x+1}$ and $y = \dfrac{x}{3} + 1$, where $0 \le x \le 3$.

d. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} 2^{n+1} x^n$.

e. Compute $\displaystyle\lim_{n\to\infty} \frac{2^n}{n!}$. *[Hint: Squeeze!]*

f. Find the volume of the solid obtained by revolving the region between $y = x$ and $y = 0$, where $0 \le x \le 1$, about the x-axis.

SOLUTIONS. **a.** Recall that the Right-Hand Rule formula is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right).$$

In our case, $a = 0$, $b = 2$, and $f(x) = x + 1$; we plug these into the formula and compute away:

$$\begin{aligned}
\int_0^2 (x+1)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2-0}{n} \left[\left(0 + i\frac{2-0}{n}\right) + 1\right] = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[i\frac{2}{n} + 1\right] \\
&= \lim_{n\to\infty} \frac{2}{n} \left[\left(\sum_{i=1}^{n} i\frac{2}{n}\right) + \left(\sum_{i=1}^{n} 1\right)\right] = \lim_{n\to\infty} \frac{2}{n} \left[\frac{2}{n}\left(\sum_{i=1}^{n} i\right) + n\right] \\
&= \lim_{n\to\infty} \frac{2}{n} \left[\frac{2}{n} \cdot \frac{n(n+1)}{2} + n\right] = \lim_{n\to\infty} \frac{2}{n} \left[(n+1) + n\right] \\
&= \lim_{n\to\infty} \frac{2}{n} \left[2n + 1\right] = \lim_{n\to\infty} \left[4 + \frac{2}{n}\right] = 4 + 0 = 4 \quad \square
\end{aligned}$$

b. The key here is the following inequality, which works for all $n \ge 2$.

$$0 < \frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{n\dot{n} \cdot n \cdots n \cdot n \cdot n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \le \frac{2}{n} \cdot \frac{1}{n} = \frac{2}{n^2}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test because $p = 2 - 0 - 2 > 1$, so it follows by the Basic Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ also converges. $\square$

c. First, note that $y = \sqrt{x+1}$ and $y = \dfrac{x}{3} + 1$ intersect at the endpoints of the given interval: $\sqrt{0+1} = 1 = \frac{0}{3} + 1$ and $\sqrt{3+1} = \sqrt{4} = 2 = \frac{3}{3} + 1$. Since the former curve is a piece of a quadratic lying on its side and the latter curve is a straight line, it's not hard to see that there can't be any more points of intersection. Second, it's also easy to check that between 0 and 3, $y = \sqrt{x+1}$ is above $y = \dfrac{x}{3} + 1$: at $x = 1$ we have $\sqrt{1+1} = \sqrt{2} \approx 1.4 > 1.3 \approx \frac{1}{3} + 1$. Thus the area between the curves for $0 \le x \le 3$ is:

$$\int_0^3 \left[\sqrt{x+1} - \left(\frac{x}{3} + 1\right)\right] dx = \int_0^3 (x+1)^{1/2}\, dx - \int_0^3 \left(\frac{x}{3} + 1\right) dx$$

In the first part we use the substitution $u = x + 1$, so $du = dx$ and $\begin{matrix} x & 0 & 3 \\ u & 1 & 4 \end{matrix}$.

$$= \int_1^4 u^{1/2}\, du - \left(\frac{x^2}{6} + x\right)\Big|_0^3 = \frac{u^{3/2}}{3/2}\Big|_1^4 - \left(\frac{x^2}{6} + x\right)\Big|_0^3$$

$$= \left[\frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2}\right] - \left[\left(\frac{3^2}{6} + 3\right) - \left(\frac{0^2}{6} + 0\right)\right]$$

$$= \left[\frac{16}{3} - \frac{2}{3}\right] - \left[\frac{9}{2} - 0\right] = \frac{14}{3} - \frac{9}{2} = \frac{28}{6} - \frac{27}{6} = \frac{1}{6} \quad \square$$

d. *(Brute Force)* As usual we first throw the Ratio Test at the given power series:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{(n+1)+1}x^{n+1}}{2^{n+1}x^n}\right| = \lim_{n\to\infty} |2x| = 2|x|$$

It follows from the Ratio Test that the power series $\sum_{n=0}^{\infty} 2^{n+1}x^n$ converges when $2|x| < 1$, *i.e.* when $|x| < \dfrac{1}{2}$, and diverges when $2|x| > 1$, *i.e.* when $|x| > \dfrac{1}{2}$. Hence the radius of convergence of the series (about $a = 0$) is $R = \dfrac{1}{2}$. To determine the interval of convergence we have to see what happens at the endpoints $x = \pm\dfrac{1}{2}$.

At $x = \dfrac{1}{2}$ the series becomes $\sum_{n=0}^{\infty} 2^{n+1}\left(\dfrac{1}{2}\right)^n = \sum_{n=0}^{\infty} 2$. This diverges by the Divergence Test because $\lim_{n\to\infty} 2 = 2 \neq 0$.

At $x = -\dfrac{1}{2}$ the series becomes $\sum_{n=0}^{\infty} 2^{n+1}\left(-\dfrac{1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n 2$. This diverges by the Divergence Test because $\lim_{n\to\infty} (-1)^n 2$ does not exist: all the odd-numbered terms ar -2 and all the even-numbered terms are 2.

Thus the interval of convergence of the given series is $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$. □

d. *(A Little Cleverness)* $\sum_{n=0}^{\infty} 2^{n+1}x^n = \sum_{n=0}^{\infty} 2(2x)^n$ is a geometric series with first term $a = 2$ and common ratio $r = 2x$. It therefore converges exactly when $|r| = |2x| < 1$, *i.e.*

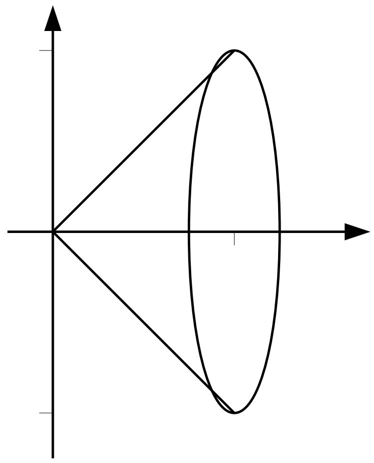
exactly when $|x| < \frac{1}{2}$, and diverges otherwise. This means that the radius of convergence is $R = \frac{1}{2}$ and the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$. □

e. The key here is the following inequality, which works for all $n \geq 2$.

$$0 < \frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2 \cdot 2}{n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1} = \frac{2}{n} \cdot \frac{2}{n-1} \cdot \frac{2}{n-2} \cdots \frac{2}{3} \cdot \frac{2}{2} \cdot \frac{1}{2} \leq \frac{2}{n} \cdot \frac{1}{2} = \frac{1}{n}$$

Since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{n}$, it follows by the Squeeze Theorem that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$, too. □

f. *(Disks)* The solid of revolution in question is a cone whose axis of symmetry is the x-axis, with the tip at the origin and the blunt end facing to the right.



We will use the disk method to compute its volume. The cross section at x of the cone is a disk with radius $r = x - 0 = x$. It follows that the volume of the solid is given by:

$$V = \int_0^1 \pi r^2 \, dx = \int_)^1 \pi x^2 \, dx = \pi \frac{x^3}{3}\Big|_0^1 = \pi\frac{1^3}{3} - \pi\frac{0^3}{3} = \frac{\pi}{3} \quad \blacksquare$$

f. *(Shells)* We will use the cylindrical shell method to find the volume of the cone. Since we revolved the triangle about the x-axis, we will use y as the native variable; note that $0 \leq y \leq 1$ for the original triangle. The cylindrical shell at y has radius $r = y - 0 = y$ and height or length $h = 1 - x = 1 - y$. t follows that the volume of the solid is given by:

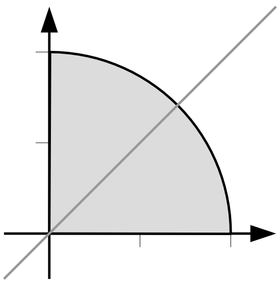
$$V = \int_0^1 2\pi r h \, dy = 2\pi \int_0^1 y(1-y)\, dy = 2\pi \int_0^1 \left(y - y^2\right) dy = 2\pi \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1$$

$$= 2\pi \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - 2\pi \left(\frac{0^2}{2} - \frac{0^3}{3}\right) = 2\pi \left(\frac{1}{2} - \frac{1}{3}\right) - 2\pi \cdot 0 = 2\pi \frac{1}{6} - 0 = \frac{\pi}{3} \quad \square$$

f. *(Geometry)* The formula for the volume of a cone with base radius r and height h is $V = \frac{\pi r^2 h}{3}$. In this case $r = h = 1$, so $V = \frac{\pi 1^2 1}{3} = \frac{\pi}{3}$. □

4. Find the centroid of the region below $y = \sqrt{4 - x^2}$ and above $y = 0$, where $0 \leq x \leq 2$. [You may assume that the density is constant and units have been chosen so that *mass* $=$ *area*.] *[12]*

SOLUTION. $y = \sqrt{4 - x^2}$, where $0 \leq x \leq 2$, is the part of the circle of radius 2 centred at the origin, *i.e.* $x^2 + y^2 = 4$, in the first quadrant (where both x and y are positive). Thus the region in question is the quarter-disk enclosed by that circle that is also in the first quadrant:



It is easy to see that the region is symmetric about the line $y = x$, so the centroid must be on this line, *i.e.* we must have $\bar{x} = \bar{y}$. We will work out $\bar{x}$ and thus get $\bar{y}$ for free.

First, the mass of the region is just the area of one quarter of a circle of radius 2, namely mass $= \frac{1}{4} \cdot \pi 2^2 = \frac{1}{4} \cdot 4\pi = \pi$.

Second, we compute the x-moment. We will use the substitution $u = 4 - x^2$, so $du = -2x\,dx$ and $x\,dx = \left(-\frac{1}{2}\right) du$, and change the limits as we go along: $\begin{matrix} x & 0 & 2 \\ u & 4 & 0 \end{matrix}$.

$$\begin{aligned}
x\text{-moment} &= \int_0^2 x \cdot (\text{length of cross-section at } x)\,dx \\
&= \int_0^2 x\left(\sqrt{4 - x^2} - 0\right) dx = \int_4^0 \sqrt{u}\left(-\frac{1}{2}\right) du \\
&= \left(-\frac{1}{2}\right)(-1)\int_0^4 u^{1/2}\,du = \frac{1}{2} \cdot \frac{u^{3/2}}{3/2}\Bigg|_0^4 = \frac{1}{3}u^{3/2}\Bigg|_0^4 \\
&= \frac{1}{3}4^{3/2} - \frac{1}{3}0^{3/2} = \frac{8}{3} - 0 = \frac{8}{3}
\end{aligned}$$

It follows that $\bar{x} = \dfrac{x\text{-moment}}{\text{mass}} = \dfrac{8/3}{\pi} = \dfrac{8}{3\pi} \approx 0.8488$. As noted above, $\bar{y} = \bar{x}$, so the centroid of the given region is the point $\left(\dfrac{8}{3\pi}, \dfrac{8}{3\pi}\right) \approx (0.8488, 0.8488)$. ■

Part Y. Do either *one* (1) of **5** or **6**. *[14]*

5. Consider the curve $y = \frac{2}{3}x^{3/2}$, where $0 \le x \le 3$.

a. Find the arc-length of the curve. *[7]*

b. Find the area of the surface obtained by revolving the curve about the y-axis. *[7]*

SOLUTIONS. **a.** Note that $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2}{3}x^{3/2}\right) = \frac{2}{3}\cdot\frac{3}{2}x^{1/2} = x^{1/2}$. We plug this into the arc-length formula and integrate away, using the substitution $u = x+1$, so $du = dx$, and change the limits as we go along: $\begin{matrix} x & 0 & 3 \\ u & 1 & 4 \end{matrix}$.

$$\begin{aligned}
\text{arc-length} &= \int_0^3 ds = \int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^3 \sqrt{1+\left(x^{1/2}\right)^2}\,dx = \int_0^3 \sqrt{1+x}\,dx \\
&= \int_1^4 \sqrt{u}\,du = \int_1^4 u^{1/2}\,du = \left.\frac{u^{3/2}}{3/2}\right|_1^4 = \left.\frac{2}{3}u^{3/2}\right|_1^4 = \frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2} \\
&= \frac{2}{3}8 - \frac{2}{3}1 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3} = 4.\dot{6} \quad \square
\end{aligned}$$

b. If the curve is revolved about the y-axis, the piece of its arc at x is revolved through a circle with radius $r = x - 0 = x$. We plug this, along with $\frac{dy}{dx} = x^{1/2}$, as obtained above, into the area formula for a surface of revolution. This will use the same substitution that was used in the solution to **a** above, with the added note that since $u = x+1$, we have $x = u - 1$.

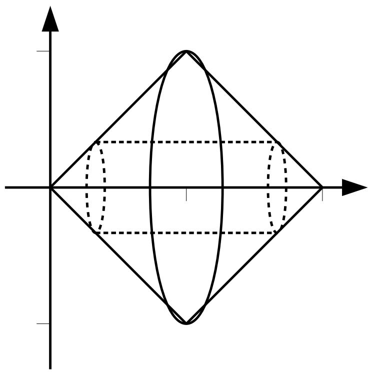
$$\begin{aligned}
\text{surface area} &= \int_0^3 2\pi r\,ds = \int_0^3 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^3 x\sqrt{1+\left(x^{1/2}\right)^2}\,dx \\
&= 2\pi\int_0^3 x\sqrt{1+x}\,dx = 2\pi\int_1^4 (u-1)\sqrt{u}\,du = 2\pi\int_1^4 (u-1)u^{1/2}\,du \\
&= 2\pi\int_1^4 \left(u^{3/2} - u^{1/2}\right)du = 2\pi\left.\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right)\right|_1^4 = 2\pi\left.\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\right|_1^4 \\
&= 2\pi\left(\frac{2}{5}4^{5/2} - \frac{2}{3}4^{3/2}\right) - 2\pi\left(\frac{2}{5}1^{5/2} - \frac{2}{3}1^{3/2}\right) \\
&= 2\pi\left(\frac{2}{5}32 - \frac{2}{3}8\right) - 2\pi\left(\frac{2}{5}1 - \frac{2}{3}1\right) = 2\pi\left(\frac{64}{5} - \frac{16}{3}\right) - 2\pi\left(\frac{2}{5} - \frac{2}{3}\right) \\
&= 2\pi\left(\frac{192-80}{15} - \frac{6-10}{15}\right) = 2\pi\frac{116}{15} = \frac{232}{15}\pi \approx 48.59 \quad \blacksquare
\end{aligned}$$

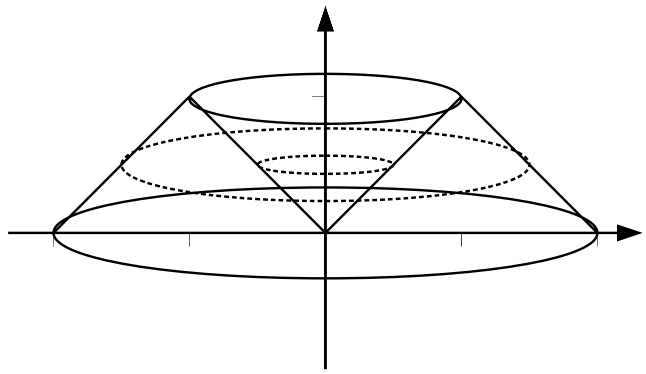
6. Consider the triangle whose vertices are the points $(0,0)$, $(1,1)$, and $(2,0)$. Find the volume of the solid obtained by revolving this triangle about ...

a. ... the x-axis. *[7]*

b. ... the y-axis. *[7]*

SOLUTIONS. First, note that the line joining $(0,0)$ to $(1,1)$ is a piece of $y = x$, the line joining $(0,0)$ to $(2,0)$ is a piece of $y = 0$, and the line joining $(1,1)$ to $(2,0)$ is a piece of $y = 2 - x$ or $x = 2 - y$, depending on how you look at it. Here are sketches of the two solids:





a. *(Shells)* We will use the method of cylindrical shells to find the volume of the solid. Since we revolved the region about the x-axis, we will use y as the variable. Note that $0 \le y \le 1$ for this region. The cylindrical shell at y has radius $r = y - 0 = y$ and height or length that is the difference of the corresponding x-values on the line $y = 2 - x$ and $y = x$, *i.e.* $h = (2 - y) - y = 2 - 2y$. We plug these into the cylindrical shell formula and integrate away:

$$V = \int_0^1 2\pi r h \, dy = \int_0^1 2\pi y(2 - 2y) \, dy = 4\pi \int_0^1 \left(y - y^2\right) \, dy = 4\pi \left(\frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_0^1$$

$$= 4\pi \left(\frac{1^2}{2} - \frac{1^3}{3} \right) - 4\pi \left(\frac{0^2}{2} - \frac{0^3}{3} \right) = 4\pi \left(\frac{1}{2} - \frac{1}{3} \right) - 0 = 4\pi \frac{1}{6} = \frac{2\pi}{3} \quad \square$$

a. *(Cunning)* The solid obtained by revolving the triangle about the x-axis is just two copies of the cone in **3f** stuck together blunt end to blunt end. It should thus have twice the volume of that cone, that is, $V = 2 \cdot \frac{\pi}{3} = \frac{2\pi}{3}$. $\square$

b. *(Washers)* We will use the washer method to find the volume of the solid. Since we revolved the region about the y-axis, we will use y as the variable. Note that $0 \leq y \leq 1$ for this region. The washer at y has outer radius $R = (2-y) - 0 = 2-y$ and inner radius – the radius of the hole – $r = y - 0 = y$. We plug these into the washer formula and integrate away:

$$
\begin{aligned}
V &= \int_0^1 \pi \left(R^2 - r^2\right)\, dy = \pi \int_0^1 \left((2-y)^2 - y^2\right)\, dy = \pi \int_0^1 \left(4 - 4y + y^2 - y^2\right)\, dy \\
&= \pi \int_0^1 (4-4y)\, dy = \pi \left(4y - 4\frac{y^2}{2}\right)\Big|_0^1 = \pi\left(4y - 2y^2\right)\Big|_0^1 \\
&= \pi\left(4 \cdot 1 - 2 \cdot 1^2\right) - \pi\left(4 \cdot 0 - 2 \cdot 0^2\right) = 2\pi - 0 = 2\pi \quad \square
\end{aligned}
$$

b. *(Cunning)* The solid obtained by revolving the triangle about the y-axis is what you get when you take a cone with twice the linear dimensions (and hence $2^3 = 8$ times the volume) of the one in **3f** and then remove two copies of the cone in **3f**. (Look at the picture and think about it ...) It should therefore have six (as $6 = 8 - 2$) times the volume of the cone in **3f**, namely $6 \cdot \frac{\pi}{3} = 2\pi$. ■

NOTE: **a** can also be done with the disk/washer method and **b** can also be done with the cylindrical shell method. This would mean using x as the variable of integration in each case; the small complication in each case is that would have to break the integral up into two pieces because the upper boundary of the original region is $y = x$ when $0 \leq x \leq 1$ and $y = 2 - x$ when $1 \leq x \leq 2$.

Part Z. Do either *one* (1) of **7** or **8**. *[14]*

7. Find the Taylor series at $a = 0$ of $f(x) = \dfrac{1}{1+2x}$ and determine its radius and interval of convergence.

SOLUTION. Recall that the sum of a geometric series $k + kz + kz^2 + kz^3 + \cdots$ with first term k and common ratio z, which converges exactly when $|z| < 1$, is $\dfrac{k}{1-z}$. The function $f(x) = \dfrac{1}{1+2x} = \dfrac{1}{1-(-2x)}$ can be thought of as the sum of such a series with $k = 1$ and $z = -2x$, which means that it equals the series $1 - 2x + (2x)^2 - (2x)^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n 2^n x^n$ when this series converges, which it does exactly when $|-2x| < 1$, *i.e.* exactly when $|x| < \dfrac{1}{2}$, which is to say $-\dfrac{1}{2} < x < \dfrac{1}{2}$. It follows that the Taylor series at $a = 0$ of $f(x) = \dfrac{1}{1+2x}$ is $1 + (-2x) + (-2x)^2 + (-2x)^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n 2^n x^n$, whose radius of convergence is $R = \dfrac{1}{2}$ and whose interval of convergence is $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$. □

8. **a.** Use Taylor's formula to find the Taylor series at $a = 0$ of $g(x) = e^x$ and determine its radius and interval of convergence. *[10]*

b. How many terms of this Taylor series are needed to guarantee that if the partial sum is evaluated at $x = 1$, it will be within $0.01 = \dfrac{1}{100}$ of $g(1) = e^1 = e$? *[4]*

SOLUTIONS. **a.** As usual, we grind out the first few derivatives of $f(x)$, evaluate them at $a = 0$, and look for a pattern giving us a general formula.

n	0	1	2	3	$\cdots$
$f^{(n)}(x)$	e^x	e^x	e^x	e^x	$\cdots$
$f^{(n)}(0)$	1	1	1	1	$\cdots$

It's not hard to conclude that $f^{(n)}(0) = 1$ for all n here. Plugging this into Taylor's formula tells us that the Taylor series of $f(x) = e^x$ at $a = 0$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

It remains to determine the radius and interval of convergence of this power series. As usual, we apply the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 < 1$$

It follows by the Ratio Test that the series converges for all x, so its radius of convergence is $R = \infty$ and its interval of convergence is $(-\infty, \infty)$. $\square$

b. Recall from class, or §11.11 in the textbook, that if $T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n$ is the Taylor polynomial of degree k – otherwise known as the kth partial sum of the Taylor series of $f(x)$ at $a = 0$ – and $R_k(x) = f(x) - T_k(x)$ is the corresponding remainder term, then $R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1}$ for some c strictly between 0 and x.*

In our case $f(1) = e^1 = e$ and the difference between this and the value of $T_k(1) = \sum_{n=0}^{k} \frac{1^n}{n!} = \sum_{n=0}^{k} \frac{1}{n!}$ is equal to $R_k(1) = \frac{f^{(k+1)}(c)}{(k+1)!} 1^{k+1} = \frac{e^c}{(k+1)!}$ for some c with $0 < c < 1$. Since e^x is an increasing function, $R_k(1) = \frac{e^c}{(k+1)!} < \frac{e^1}{(k+1)!} < \frac{3}{(k+1)!}$, so all we need to do is find a k large enough to ensure that $\frac{3}{(k+1)!} < 0.01 = \frac{1}{100}$. Cross-multiplying, this boils down finding the first k such that $300 < (k+1)!$:

k	0	1	2	3	4	5	$\cdots$
$(k+1)!$	1	2	6	24	120	720	$\cdots$

The first such k is therefore $k = 5$. It follows that $T_k(1) = \sum_{n=0}^{k} \frac{1}{n!}$ is guaranteed to be within 0.01 of e whenever $k \geq 5$. Since $T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n$ includes the first $k+1$ terms of the Taylor series of $f(x)$ at $a = 0$, it follows that adding up at least $5 + 1 = 6$ terms of the series suffices to compute e to within 0.01. $\blacksquare$

[Total = 100]

* This is the Lagrange form of the remainder. Note that all this supposes that x is within the radius of convergence of the Taylor series of $f(x)$ at $a = 0$.

Part W. Bonus problems! If you feel like it and have the time, do one or both of these.

W. Consider the following real number:

$$a = \sum_{n=0}^{\infty} \frac{1}{10^{[2^n]}} = \sum_{n=0}^{\infty} 10^{-[2^n]} = 0.11010001000000010\cdots$$

[For $k \geq 1$, there are $2^{k-1} - 1$ zeros between the kth and $(k+1)$st ones in the decimal expansion of a.] Explain why a must be irrational. *[1]*

Solution. If a real number is rational, then past some point its decimal expansion will repeat some finite sequence of digits forever. For example, $\frac{1}{3} = 0.\dot{3} = 0.333333\cdots$ repeats the sequence "3", while $\frac{31}{70} = 0.4\overline{428571} = 0.4428571428571428571\cdots$ repeats the sequence "428571". Since the number a above has an increasing number of zeros between successive ones in its decimal expansion, there is no point at which it will begin to repeat some finite sequence of digits forever, and thus it can't be rational. Hence, by definition, it must be irrational. ■

M. Write a haiku (or several :-) touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

Solution. Here is another of your instructor's haiku:

What is mathematics?

The art of drawing
necessary conclusions
about abstract things.

That's all, folks! ■

Enjoy your summer!