**Mathematics 1120H – Calculus II: Integrals and Series**
Trent University, Winter 2019

**Assignment #3**
**Series, inverse squares, and trig**
*Due on Friday, 15 February.*

Your task on this assignment will be to show that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

**1.** Verify the following trigonometric identity. (So long as $x$ is not an integer multiple of $\pi$ anyway! :-) *[2]*

$$\frac{1}{\sin^2(x)} = \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{\sin^2\left(\frac{x+\pi}{2}\right)}\right]$$

*Hint:* Use common trig identities and the fact that for any $t$, $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$.

**2.** Verify the following trigonometric summation formula for $m \geq 1$. *[2]*

$$1 = \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)}$$

*Hint:* Apply the identity from question **1** repeatedly, starting from $1 = \dfrac{1}{\sin^2\left(\frac{\pi}{2}\right)}$. You may find the fact that $\sin(t) = \sin(\pi - t)$ comes in handy.

**3.** Verify the following limit formula, where $k \geq 0$ is fixed. *[2]*

$$\lim_{m\to\infty} 2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right) = \frac{(2k+1)\pi}{2}$$

*Hint:* This is really just (a version of) $\lim_{t\to 0} \frac{\sin(t)}{t} = 0 \ldots$

**4.** Take the limit as $m \to \infty$ of the identity in **2**, and use **3** to show the following. *[2]*

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

Note: Here you will need to interchange a limit with a sum, which you may do without having to justify it. (That's the one thing in this argument that is not really first-year-calculus-level material.)

**5.** Use **4** and some algebra to check that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

is true. *[2]*

*Hint:* Split up $\sum_{n=1}^{\infty} \frac{1}{n^2}$ into the sums of the terms for even and odd $n$ respectively and try to rewrite the sum of the terms for even $n$.