Mathematics 1120H – Calculus II: Integrals and Series

Trent University, Winter 2019

Assignment #1

A little bit about the Riemann integral

Due on Friday, 18 January.

Recall from MATH 11120H [especially Assignment #9], class, or the textbook, that the definite integral $\int_a^b f(x)\,dx$ essentially gives the signed or weighted area of the region between $y = f(x)$ and the x-axis, where area above the x-axis is added and area below the x-axis is subtracted. The definite integral is usually defined in terms of limits of Riemann sums, but the full general definition is pretty cumbersome to work with. This assignment is meant to give you a little bit of practice with it and give an inkling as to why simplifications like the Right-Hand Rule are not quite enough to justify all the properties of definite integrals.

First, here is the aforementioned Right-Hand Rule, which will, in principle, properly compute $\int_a^b f(x)\,dx$ for most commonly encountered functions.

Right-Hand Rule. Suppose $f(x)$ is defined for all x in $[a, b]$ and is continuous at all but finitely many points of $[a, b]$. Then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=0}^{n}\frac{b-a}{n}f\left(a + i\cdot\frac{b-a}{n}\right)\right]$$

The idea here is that we divide up the interval $[a, b]$ into n subintervals of equal width $\frac{b-a}{n}$, so the ith subinterval, going from left to right and where $1 \le i \le n$, will be $\left[(i-1)\cdot\frac{b-a}{n}, i\cdot\frac{b-a}{n}\right]$. Each subinterval serves as the base of a rectangle of height $f\left(a + i\cdot\frac{b-a}{n}\right)$, which must then have area $\frac{b-a}{n}f\left(a + i\cdot\frac{b-a}{n}\right)$. The sum of the areas of these rectangles, the *nth Right-Hand Rule sum for* $\int_a^b f(x)\,dx$, approximates the area computed by $\int_a^b f(x)\,dx$. (It's called the Right-Hand Rule because it uses the right-hand endpoint of each subinterval to evaluate $f(x)$ at to determine the height of the rectangle which has that subinterval as a base.) As we increase n and so shrink the width of the rectangles we get better and better approximations to the definite integral.

1. Use the Right-Hand Rule to show that $\displaystyle\int_0^a e^x\,dx = e^a$. *[3]*

Hint. You'll need to do some algebra and may want to look up geometric series (Example 11.2.1 in the textbook) and their summation formulas if you don't remember them.

2. Suppose $f(x)$ is a function which is defined and continuous – and hence ought to be integrable – on all of $\mathbb{R}$. Explain why we would have a problem justifying

$$\int_{-1}^0 f(x)\,dx + \int_0^{\sqrt{2}} f(x)\,dx = \int_{-1}^{\sqrt{2}} f(x)\,dx$$

if we used the Right-Hand Rule (or any rule that relies on subdividing $[a, b]$ into equal subintervals) as the actual definition of $\int_a^b f(x)\,dx$. *[3]*

Hint. It matters here that $\sqrt{2}$ is irrational.

More on page 2!

The full definition of the Riemann integral uses the same basic idea of approximating the area under a curve by rectangles, but it allows for these rectangles to have differing widths and heights that are computed by evaluating the function at arbitrary points in the bases of the rectangles. This, sadly, means that we need some preliminary definitions and notation.

PARTITIONS. If $[a, b]$ is a (closed finite) interval, then a *partition* of $[a, b]$ is a set of subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\dots$, $[x_{n-1}, x_n]$, of $[a, b]$ where $n \geq 1$ and $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. (Note that different partitions may divide up $[a, b]$ into different numbers of subintervals n.) For the sake of brevity, we will often denote such a partition by $\{x_i\}$.

If $\{x_i\}$ is a partition of $[a, b]$, its *norm* $\|\{x_i\}\|$ is the maximum width of a subinterval of the partition, *i.e.* $\|\{x_i\}\| = \max\{\, x_i - x_{i-1} \mid 1 \leq i \leq n \,\}$.

A *tagged partition* of $[a, b]$ is a partition $\{x_i\}$ of $[a, b]$ together with some choice of points x_i^* for each i with $1 \leq i \leq n$, where the only restriction is that $x_{i-1} \leq x_i^* \leq x_i$ for each i. For the sake of brevity, we will often denote a tagged partition by $\{x_i\}^*$

SUPREMA. If A is a set of real numbers with an *upper bound* u, *i.e.* $a \leq u$ for all $a \in A$, then A has a *least upper bound* or *supremum*, often denoted by $\sup(A)$, which is an upper bound of A and such that $\sup(A) \leq u$ for every other upper bound u of A. Note that if A is finite, then $\sup(A) = \max\{\, a \mid a \in A \,\} \in A$, but this doesn't have to be true if A is infinite. For example, $\sup\left((0, 1)\right) = 1 \notin (0, 1)$.

THE RIEMANN INTEGRAL. If $f(x)$ is a function defined on $[a, b]$ and $\{x_i\}^*$ is a tagged partition of $[a, b]$, the corresponding *Riemann sum* is $R\left(f, \{x_i\}^*\right) = \displaystyle\sum_{i=1}^{n} (x_i - x_{i-1})\, f\left(x_i^*\right)$. This is just the sum of the areas of n rectangles, where the base of the ith rectangle is the subinterval $[x_{i-1}, x_i]$ and its height is $f\left(x_i^*\right)$. Note that Right-Hand Rule sums are Riemann sums for certain very special tagged partitions.

If $f(x)$ is a function defined on $[a, b]$, then its *Riemann integral* on $[a, b]$ is

$$\int_a^b f(x)\, dx = \lim_{\delta \to 0^+} \sup\left(\left\{\, R\left(f, \{x_i\}^*\right) \mid \|\{x_i\}\| < \delta \,\right\}\right) ,$$

provided that this limit exists. If the limit does exist, then $f(x)$ is said to be *Riemann integrable* on $[a, b]$. Actually working with this definition is usually pretty hard because the definitional pile-up leading up to it means that the limit is very complicated. To make it worse, in practice you often have to go to the ε–δ definition of limits when working with it, too.

3. Suppose $f(x)$ is a function which is Riemann integrable on $\mathbb{R}$. Explain why the definition of the Riemann integral does justify

$$\int_{-1}^{0} f(x)\, dx + \int_{0}^{\sqrt{2}} f(x)\, dx = \int_{-1}^{\sqrt{2}} f(x)\, dx \,.$$

You explanation should be informal, but as complete and precise as you can. *[3]*

0. Please give your name! No name, no mark ... *[1]*