Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals

TRENT UNIVERSITY, Summer 2018

MATH 1120H Practice Test Solutions

Time: 50 minutes

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and (all sides of) an aid sheet.

1. Compute any *four* (4) of integrals **a**–**f**. *[12 = 4 × 3 each]*

a. $\displaystyle\int \tan^2(x)\cos^3(x)\,dx$ **b.** $\displaystyle\int_2^\infty \frac{1}{y^3}\,dy$ **c.** $\displaystyle\int \frac{z^2+1}{z^2-1}\,dz$

d. $\displaystyle\int_0^{\ln(10)} te^{-t}\,dt$ **e.** $\displaystyle\int \frac{1}{2\sqrt{s}\cdot\ln\left(\sqrt{s}\right)}\,ds$ **f.** $\displaystyle\int_0^3 \frac{1}{r^2+9}\,dr$

SOLUTIONS. **a.** After some preliminary algebra, we will use the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$.

$$\int \tan^2(x)\cos^3(x)\,dx = \int \left(\frac{\sin(x)}{\cos(x)}\right)^2 \cos^3(x)\,dx = \int \frac{\sin^2(x)}{\cos^2(x)}\cos^3(x)\,dx$$
$$= \int \sin^2(x)\cos(x)\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3(x) + C \quad \square$$

b. This is obviously an improper integral:

$$\int_2^\infty \frac{1}{y^3}\,dy = \lim_{t\to\infty}\int_2^t \frac{1}{y^3}\,dy = \lim_{t\to\infty}\int_2^t y^{-3}\,dy = \lim_{t\to\infty}\left.\frac{y^{-2}}{-2}\right|_2^t = \lim_{t\to\infty}\left.\frac{-1}{2y^2}\right|_2^t$$
$$= \lim_{t\to\infty}\left[\left(-\frac{1}{2t^2}\right) - \left(-\frac{1}{2\cdot 2^2}\right)\right] = \lim_{t\to\infty}\left[-\frac{1}{2t^2} + \frac{1}{8}\right] = 0 + \frac{1}{8} = \frac{1}{8} \quad \square$$

c. $\displaystyle\int \frac{z^2+1}{z^2-1}\,dz = \int \frac{z^2-1+2}{z^2-1}\,dz = \int \left(\frac{z^2-1}{z^2-1} + \frac{2}{z^2-1}\right)dz = \int 1\,dz + \int \frac{2}{z^2-1}\,dz.$
The first integral is easy, to do the second we need to apply the method of partial fractions. $z^2-1 = (z-1)(z+1)$, so, for some constants A and B, we have $\dfrac{2}{z^2-1} = \dfrac{A}{z-1} + \dfrac{B}{z+1} = \dfrac{A(z+1)+B(z-1)}{(z-1)(z+1)} = \dfrac{(A+B)z+(-A+B)}{z^2-1}$. It follows that $A+B=0$ and $-A+B=2$. Adding the two equations gives us $2B = 2$, so $B = 1$, and it then follows from the first equation that $A = -B = -1$. Thus:

$$\int \frac{z^2+1}{z^2-1}\,dz = \int 1\,dz + \int \frac{2}{z^2-1}\,dz = z + \int \frac{-1}{z-1}\,dz + \int \frac{1}{z+1}\,dz$$
$$= z - \ln(z-1) + \ln(z+1) + C \quad \square$$

d. We will use integration by parts, with $u = t$ and $v' = e^{-t}$, so $u' = 1$ and $v = (-1)e^{-t}$.

$$\begin{aligned}
\int_0^{\ln(10)} te^{-t}\,dt &= t\cdot(-1)e^{-t}\Big|_0^{\ln(10)} - \int_0^{\ln(10)} 1\cdot(-1)e^{-t}\,dt \\
&= (-1)te^{-t}\Big|_0^{\ln(10)} - (-1)(-1)e^{-t}\Big|_0^{\ln(10)} \\
&= \left[(-1)\ln(10)e^{-\ln(10)} - (-1)\cdot 0\cdot e^{-0}\right] - \left[e^{-\ln(10)} - e^{-0}\right] \\
&= \left[\frac{(-1)\ln(10)}{e^{\ln(10)}} + 0\right] - \left[\frac{1}{e^{\ln(10)}} - 1\right] = -\frac{\ln(10)}{10} - \frac{1}{10} + 1 \\
&= \frac{9}{10} - \frac{\ln(10)}{10} = \frac{9-\ln(10)}{10} \approx 0.6697 \quad \square
\end{aligned}$$

e. We'll use the substitution $w = \sqrt{s}$, so $dw = \frac{1}{2\sqrt{s}}\,ds$. Then

$$\int \frac{1}{2\sqrt{s}\cdot\ln\left(\sqrt{s}\right)}\,ds = \int \frac{1}{\ln(w)}\,dw\,,$$

which leaves the problem if finding the antiderivative of $\dfrac{1}{\ln(w)}$. At this point there are two reasonable ways to proceed with what we know so far in the course. (Once we have Taylor series in hand, there will be another potential option.)

First, one could try using integration by parts with $u = \dfrac{1}{\ln(w)}$ and $v' = 1$, so $u' = \dfrac{-1}{[\ln(w)]^2}\cdot\dfrac{1}{w}$ and $v = w$. Once you plug this in, though, you get

$$\int \frac{1}{\ln(w)}\,dw = \frac{1}{\ln(w)}\cdot w - \int \frac{-1}{[\ln(w)]^2}\cdot\frac{1}{w}\cdot w\,dw = \frac{w}{\ln(w)} + \int \frac{1}{[\ln(w)]^2}\,dw\,,$$

which is pretty clearly not an improvement. If one continues with this approach, one does eventually get the fascinating formula $\displaystyle\int \frac{1}{\ln(w)}\,dw = w\sum_{n=1}^{\infty}\frac{(n-1)!}{[\ln(w)]^n}$, but we haven't gotten to power series yet, much less series of functions in general.

Second, one could try the substitution $w = e^t$, so $dw = e^t\,dt$, to get rid of the natural logarithm:

$$\int \frac{1}{\ln(w)}\,dw = \int \frac{1}{\ln\left(e^t\right)}\,e^t\,dt = \int \frac{e^t}{t}\,dt = \int t^{-1}e^t\,dt$$

This integral, sadly, is about as difficult to evaluate as the original. For example, if one tries to use integration by parts in the obvious way, with $u = t^{-1}$ and $v' = e^t$ [you figure out why the alternatives are worse!], so $u' = (-1)t^{-2}$ and $v = e^t$, one gets:

$$\int t^{-1}e^t\,dt = t^{-1}e^t - \int (-1)t^{-2}e^t\,dt = t^{-1}e^t + \int t^{-2}e^t\,dt$$

This isn't an improvement either. This approach leads to the formula $\int t^{-1} e^t \, dt = e^t \sum_{n=1}^{\infty} (n-1)! t^{-n}$, which is sort of a power series except for all the negative exponents and that factor of e^t.

The bottom line is that this problem is way too hard for this course, which is why it won't be on the actual test. :-) □

f. We will use the substitution $r = 3v$, so $dr = 3\,dv$ and $\begin{matrix} r & 0 & 3 \\ v & 0 & 1 \end{matrix}$.

$$\begin{aligned} \int_0^3 \frac{1}{r^2+9}\,dr &= \int_0^1 \frac{1}{(3v)^2+9}\,3\,dv = \int_0^1 \frac{3}{9v^2+9}\,dv = \frac{3}{9}\int_0^1 \frac{1}{v^2+1}\,dv \\ &= \frac{1}{3}\arctan(v)\Big|_0^1 = \frac{1}{3}\arctan(1) - \frac{1}{3}\arctan(0) = \frac{1}{3}\cdot\frac{\pi}{4} - \frac{1}{3}\cdot 0 = \frac{\pi}{12} \quad \blacksquare \end{aligned}$$

2. Do any *two* (2) of parts **a**–**c**. *[8 = 2 × 4 each]*

a. Suppose $f(x) = \int_0^x e^{t^2}\,dt$. Find the antiderivative of $g(x) = xf'(x)$.

b. Find the area between the curves $y = 1 - \sqrt{x}$ and $y = 1 - x^2$ for $0 \le x \le 1$.

c. Express $\int \sec^5(x)\,dx$ in terms of $\int \sec^3(x)\,dx$.

Solutions. **a.** By the Fundamental Theorem of Calculus, $f'(x) = e^{x^2}$. Thus:

$$\begin{aligned} \int g(x)\,dx &= \int xf'(x)\,dx = \int xe^{x^2}\,dx \qquad \text{Substitute } u = x^2, \text{ so } du = 2x\,dx \text{ and } x\,dx = \tfrac{1}{2}\,du. \\ &= \int e^u \frac{1}{2}\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C \quad \square \end{aligned}$$

b. Since $\sqrt{x} \ge x^2$ for $0 \le x \le 1$, we have $1 - \sqrt{x} \le 1 - x^2$ for $0 \le x \le 1$. It follows that the given area is given by:

$$\begin{aligned} \text{Area} &= \int_0^1 \left[\left(1-x^2\right) - \left(1-\sqrt{x}\right)\right]\,dx = \int_0^1 \left[-x^2 + \sqrt{x}\right]\,dx = \int_0^1 \left[x^{1/2} - x^2\right]\,dx \\ &= \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]\Bigg|_0^1 = \left[\frac{2}{3}\cdot 1^{3/2} - \frac{1}{3}\cdot 1^3\right] - \left[\frac{2}{3}\cdot 0^{3/2} - \frac{1}{3}\cdot 0^3\right] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \quad \square \end{aligned}$$

c. We'll use integration by parts with $u = \sec^3(x)$ and $v' = \sec^2(x)$, so $u' = 3\sec^2(x)\cdot\sec(x)\tan(x) = 3\sec^3(x)\tan(x)$ and $v = \tan(x)$, as well as the trigonometric identity

$\tan^2(x) = \sec^2(x) - 1$ and some algebra.

$$\begin{aligned}\int \sec^5(x)\,dx &= \sec^3(x)\cdot\tan(x) - \int 3\sec^3(x)\tan(x)\cdot\tan(x)\,dx \\ &= \sec^3(x)\tan(x) - 3\int \sec^3(x)\tan^2(x)\,dx \\ &= \sec^3(x)\tan(x) - 3\int \sec^3(x)\left(\sec^2(x)-1\right)\,dx \\ &= \sec^3(x)\tan(x) - 3\int \sec^5(x)\,dx + 3\int \sec^3(x)\,dx\end{aligned}$$

It follows that $4\int \sec^5(x)\,dx = \sec^3(x)\tan(x) + 3\int \sec^3(x)\,dx$, and so $\int \sec^5(x)\,dx = \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{4}\int \sec^3(x)\,dx$. □

3. Do either *one* (1) of parts **a** or **b**. *[10]*

a. Use the fact that $\int_0^1 \frac{1}{x^2+1}\,dx = \frac{\pi}{4}$ and one of the Right-Hand Rule or the Trapezoid Rule to estimate π to within $0.4 = \frac{4}{10}$.

b. A spike has four triangular faces. Three of these faces are right-angled triangles, one with short sides of 1 *cm* each, and two which each have a short side of 1 *cm* and a short side of 10 *cm*. These triangles fit together so that the right-angle vertices coincide and the short sides match up in length. The fourth face is the triangle formed by the hypotenuses (hypotesunoi?) of the other three triangles. What is the volume of the spike?

SOLUTIONS. **a.** As a sanity check: $\int_0^1 \frac{1}{x^2+1}\,dx = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

We will do this using the Trapezoid Rule. (Using the Right-Hand Rule, we would have less work to do to find an upper bound for the error, but then have to evaluate a considerably longer sum.) Observe that we can estimate π to within 0.4 if we can estimate $\frac{\pi}{4} = \int_0^1 \frac{1}{x^2+1}\,dx$ to within $\frac{0.4}{4} = 0.1$. Note that, using the Quotient and Power Rules for derivatives:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x^2+1}\right) &= \frac{0\cdot\left(x^2+1\right) - 1\cdot(2x+0)}{\left(x^2+1\right)^2} = \frac{-2x}{\left(x^2+1\right)^2} \\ \frac{d^2}{dx^2}\left(\frac{1}{x^2+1}\right) &= \frac{d}{dx}\left(\frac{-2x}{\left(x^2+1\right)^2}\right) = \frac{(-2)\cdot\left(x^2+1\right)^2 - (-2x)\cdot 2\left(x^2+1\right)(2x+0)}{\left(x^2+1\right)^4} \\ &= \frac{(-2)\cdot\left(x^2+1\right) + (2x)\cdot 2(2x)}{\left(x^2+1\right)^3} = \frac{6x^2-2}{\left(x^2+1\right)^3}\end{aligned}$$

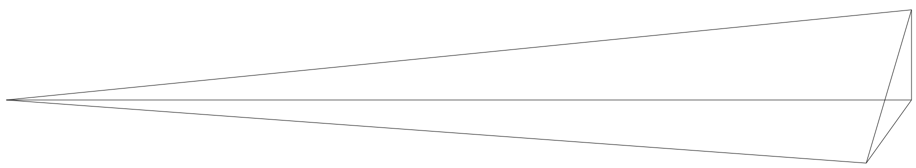
Suppose we have $0 \le x \le 1$. Then $\left(x^2+1\right)^3 \ge \left(0^2+1\right)^3 = 1^3 = 1$, so $\dfrac{1}{\left(x^2+1\right)^3} \le \dfrac{1}{1} = 1$, and $\left|6x^2-2\right| \le 6 \cdot 1^2 - 2 = 4$. Thus $\left|\dfrac{d^2}{dx^2}\left(\dfrac{1}{x^2+1}\right)\right| = \left|\dfrac{6x^2-2}{\left(x^2+1\right)^3}\right| \le 1 \cdot 4 = 4$ for $0 \le x \le 1$. It follows that $\frac{M(b-a)^3}{n^2} = \frac{4(1-0)^3}{n^2} = \frac{4}{n^2}$ is an upper bound for the difference between the Trapezoid Rule sum for n for our integral and the exact value of the integral.

We need to get this difference to be no more than 0.1: $\dfrac{4}{n^2} \le 0.1$ exactly when $40 = \dfrac{4}{0.1} \le n^2$. The first integer n for which this is true is $n = 7$, because $6^2 = 36 < 40$ and $7^2 = 49 > 40$. This means that we divide the interval $[0,1]$ into sevenths, with $x_0 = 0$, $x_1 = \frac{1}{7}$, $x_2 = \frac{2}{7}$, ... $x_7 = \frac{7}{7} = 1$. The Trapezoid Rule sum for $n = 7$ for the integral $\displaystyle\int_0^1 \frac{1}{x^2+1}\,dx$ is then (suppressing much arithmetic in evaluating the integrand at the points in question):

$$\begin{aligned}
\text{Sum} &= \left(\frac{f(0)}{2} + f\left(\frac{1}{7}\right) + \cdots + f\left(\frac{6}{7}\right) + \frac{f(1)}{2}\right) \cdot \frac{1}{7} \quad \text{(See §8.6 of the text.)} \\
&= \frac{1}{7}\left(\frac{1}{2} + \frac{49}{50} + \frac{49}{53} + \frac{49}{58} + \frac{49}{65} + \frac{49}{74} + \frac{49}{85} + \frac{49}{196}\right) \\
&= \frac{1}{14} + \frac{7}{50} + \frac{7}{53} + \frac{7}{58} + \frac{7}{65} + \frac{7}{74} + \frac{7}{85} + \frac{1}{28} \\
&= \ldots \text{ and we curse arithmetic!}
\end{aligned}$$

I hope it's obvious why this one didn't make it to the actual test! The Right-Hand Rule sum is considerably longer ... □

b. Here is a sketch of the spike, which will hopefully help with visualizing what is going on in the solution:



Consider the cross-sections of the spike that are parallel to the small triangle that is the blunt end of the spike. If x is the distance of this cross-section from the sharp end of the spike, it is not hard to see [similar triangles!] that the cross-section is a right-angled triangle with both short sides of length $\dfrac{x}{10}$ cm, and hence area $A(x) = \dfrac{1}{2}\left(\dfrac{x}{10}\right)^2 = \dfrac{x^2}{200}$. The volume of the spike is obtained by integrating the areas of the cross-sections:

$$\text{Volume} = \int_0^{10} A(x)\,dx = \int_0^{10} \frac{x^2}{200}\,dx = \frac{1}{200} \cdot \left.\frac{x^3}{3}\right|_0^{10} = \frac{10^3}{600} - \frac{0^3}{600} = \frac{1000}{600} = \frac{5}{3}\ cm^3 \quad \blacksquare$$

[Total = 30]