Mathematics 1120H – Calculus II: Integrals and Series

TRENT UNIVERSITY, Summer 2018

Solutions to Assignment #1
The Area of an Ellipse

The area of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is πab. Your task will be to verify that this is so.

1. Verify the area formula above using calculus. *[6]*

NOTE. You may need to look up an appropriate integration technique.

SOLUTION. First, observe that we may assume that $a>0$ and $b>0$: neither a nor b can be 0 for the given equation to make sense, and since they only appear as squares, we may as well replace each by its positive equivalent if it happens to be negative.

Solve for y as a function of x:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \implies y^2=b^2\left(1-\frac{x^2}{a^2}\right) \implies y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

Note that this makes sense for x with $-a\le x\le a$, and that $+b\sqrt{1-\frac{x^2}{a^2}}\ge -b\sqrt{1-\frac{x^2}{a^2}}$ for x in this range. The area of the ellipse is the area between these two curves, *i.e.* the area between the upper and lower halves of the ellipse:

$$\text{Area}=\int_{-a}^{a}\left[\left(+b\sqrt{1-\frac{x^2}{a^2}}\right)-\left(-b\sqrt{1-\frac{x^2}{a^2}}\right)\right]dx=\int_{-a}^{a}2b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

To evaluate this integral, we will use the mostly trigonometric subsitution $x=a\sin(\theta)$, so $dx=a\cos(\theta)\,d\theta$. Note that $a\sin(\theta)=-a$ when $\theta=-\frac{\pi}{2}$ and $a\sin(\theta)=a$ when $\theta=\frac{\pi}{2}$, so $\begin{matrix} x & -a & a \\ \theta & -\pi/2 & \pi/2 \end{matrix}$.

$$\begin{aligned}\text{Area}&=\int_{-a}^{a}2b\sqrt{1-\frac{x^2}{a^2}}\,dx=2b\int_{-\pi/2}^{\pi/2}\sqrt{1-\frac{a^2\sin^2(\theta)}{a^2}}\cdot a\cos(\theta)\,d\theta\\
&=2ab\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2(\theta)}\cdot\cos(\theta)\,d\theta=2ab\int_{-\pi/2}^{\pi/2}\sqrt{\cos^2(\theta)}\cdot\cos(\theta)\,d\theta\\
&=2ab\int_{-\pi/2}^{\pi/2}\cos^2(\theta)\,d\theta=2ab\int_{-\pi/2}^{\pi/2}\frac{1}{2}\left(1+\cos(2\theta)\right)\,d\theta\\
&=ab\int_{-\pi/2}^{\pi/2}\left(1+\cos(2\theta)\right)\,d\theta\end{aligned}$$

To evaluate this integral in its turn, we will use the substitution $u=2\theta$, so $du=2\,d\theta$ and

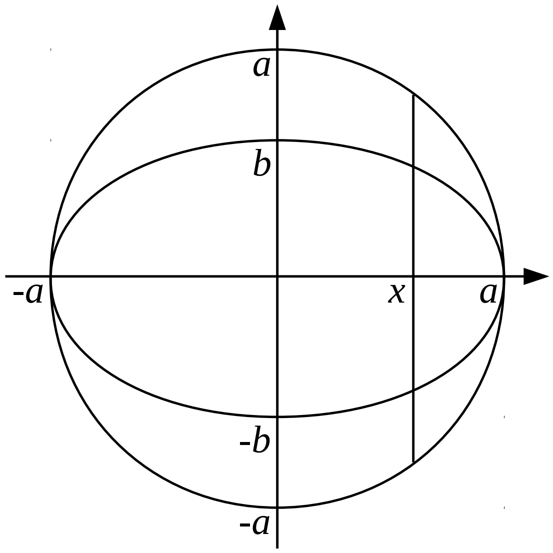
$d\theta = \frac{1}{2}\,du$, and $\begin{array}{ccc} \theta & -\pi/2 & pi/2 \\ u & -\pi & \pi \end{array}$.

$$\begin{aligned}
\text{Area} &= ab\int_{-\pi/2}^{\pi/2} (1+\cos(2\theta))\;d\theta = ab\int_{-\pi}^{\pi} (1+\cos(u))\;\frac{1}{2}\,du \\
&= \frac{ab}{2}\,(u+\sin(u))\Big|_{-\pi}^{\pi} = \frac{ab}{2}\,(\pi+\sin(\pi)) - \frac{ab}{2}\,(-\pi+\sin(-\pi)) \\
&= \frac{ab}{2}(\pi+0) - \frac{ab}{2}(-\pi+0) = \frac{ab}{2}(\pi+\pi) = \pi ab \quad \blacksquare
\end{aligned}$$

2. Verify the area formula above without using calculus. *[4]*

Hint. Compare cross-sections of the ellipse to those of a suitable circle.

SOLUTION. Following the hint, suppose we fit the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ inside the circle $x^2 + y^2 = a^2$.



(This implicitly assumes that $a \geq b$. If $b < a$, place the ellipse inside a circle of radius b instead, and use horizontal instead of vertical cross-sections in what follows.)

Consider the vertical cross-sections at x. The length of the cross-section of the ellipse at x is is $+b\sqrt{1-\frac{x^2}{a^2}} - \left(-b\sqrt{1-\frac{x^2}{a^2}}\right) = 2b\sqrt{1-\frac{x^2}{a^2}}$, and the length of the cross-section of the circle at x is $+\sqrt{a^2-x^2} - \left(-\sqrt{a^2-x^2}\right) = 2\sqrt{a^2-x^2} = 2a\sqrt{1-\frac{x^2}{a^2}}$. It follows that for each x with $-a \leq x \leq a$, we have a constant (!) ratio of lengths of vertical cross-sections of ellipse to circle of $\frac{b}{a}$; this, in turn, means that the areas swept out by the cross-sections from left to right, *i.e.* the areas enclosed by the curves, are in the ratio $\frac{b}{a}$. Since the area of the circle is πa^2, the area of the ellipse must be $\frac{b}{a}\pi a^2 = \pi ab$. $\square$

NOTE. The method used above is usually called *Cavalieri's Principle* nowadays, after the Italian mathematician Bonaventura Cavalieri (1598–1647 A.D.), who articulated and used it in a pretty general form, but the basic idea goes back at least as far as the ancient Greek mathematician Archimedes (*c.* 287 – *c.* 212 B.C.).