Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Summer 2018

Solutions to the Quizzes

Quiz #1. Wednesday, 20 June. *[20 minutes]*

Compute each of the following integrals.

1. $\displaystyle\int \sec^2(x)\tan^2(x)\,dx$ *[1]* 2. $\displaystyle\int \sec^4(x)\,dx$ *[1.5]* 3. $\displaystyle\int \sin^2(x)\cos^2(x)\,dx$ *[2.5]*

Solutions. 1. We'll use the substitution $u = \tan(x)$, in which case $\dfrac{du}{dx} = \sec^2(x)$ and $du = \sec^2(x)\,dx$:

$$\int \sec^2(x)\tan^2(x)\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{\tan^3(x)}{3} + C \quad \square$$

2. We'll use the trigonometeric identity $\tan^2(x) + 1 = \sec^2(x)$ followed by the same subsitution used in the solution to question 1 above:

$$\begin{aligned}\int \sec^4(x)\,dx &= \int \left(\tan^2(x)+1\right)\sec^2(x)\,dx = \int \left(u^2+1\right)\,du \\ &= \frac{u^3}{3} + u + C = \frac{\tan^3(x)}{3} + \tan(x) + C \quad \square\end{aligned}$$

3. *(Using the double angle formulas and a little substitution.)* We'll use the trigonometric identities $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$, with a little bit of rearranging to give $\sin(\theta)\cos(\theta) = \dfrac{1}{2}\sin(2\theta)$ and $\sin^2(\alpha) = \dfrac{1}{2}(1-\cos(2\alpha))$, plus the substitution $w = 4x$, so $dw = 4\,dx$ and $dx = \frac{1}{4}\,dw$.

$$\begin{aligned}\int \sin^2(x)\cos^2(x)\,dx &= \int (\sin(x)\cos(x))^2\,dx = \int \left(\frac{1}{2}\sin(2x)\right)^2\,dx \\ &= \frac{1}{4}\int \sin^2(2x)\,dx = \frac{1}{4}\int \frac{1}{2}(1-\cos(4x))\,dx \\ &= \frac{1}{8}\int (1 - cos(w))\,\frac{1}{4}\,dw = \frac{1}{32}(w - sin(w)) + C \\ &= \frac{1}{32}(4x - \sin(4x)) + C = \frac{1}{32}(4x - 2\sin(2x)\cos(2x)) + C \\ &= \frac{1}{32}\left(4x - 4\sin(x)\cos(x)\left(1 - 2\sin^2(x)\right)\right) + C \\ &= \frac{1}{8}\left(x - \sin(x)\cos(x) + 2\sin^3(x)\cos(x)\right) + C\end{aligned}$$

Stopping at $\dfrac{1}{32}(4x - \sin(4x)) + C$ would probably be enough for most purposes. $\square$

3. *(Using mainly integration by parts.)* We will use integration parts with $u = \cos(x)$ and $v' = \sin^2(x)\cos(x)$, so $u' = \frac{d}{dx}\cos(x) = -\sin(x)$ and

$$v = \int \sin^2(x)\cos(x)\,dx = \int w^2\,dw = \frac{1}{3}w^3 = \frac{1}{3}\sin^3(x)\,,$$

the latter calculation using the substitution $w = \sin(x)$, so $dw = \cos(x)\,dx$. It now follows that:

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= uv - \int u'v\,dx = \cos(x)\cdot\frac{1}{3}\sin^3(x) - \int (-\sin(x))\cdot\frac{1}{3}\sin^3(x)\,dx \\
&= \frac{1}{3}\cos(x)\sin^3(x) + \frac{1}{3}\int \sin^4\;dx \\
&= \frac{1}{3}\cos(x)\sin^3(x) + \frac{1}{3}\int \sin^2(x)\left(1-\cos^2(x)\right)\;dx \\
&= \frac{1}{3}\cos(x)\sin^3(x) + \frac{1}{3}\int \sin^2(x)\,dx - \frac{1}{3}\int \sin^2(x)\cos^(x)\,dx
\end{aligned}$$

A little rearranging gives $\frac{4}{3}\int \sin^2(x)\cos^2(x)\,dx = \frac{1}{3}\cos(x)\sin^3(x) + \frac{1}{3}\int \sin^2(x)\,dx$, thus, using the formula $\sin^2(\alpha) = \frac{1}{2}(1-\cos(2\alpha))$ and the substitution $w = 2x$ (so $dw = 2\,dx$ and $dx\frac{1}{2}\,dw$):

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= \frac{3}{4}\left[\frac{1}{3}\cos(x)\sin^3(x) + \frac{1}{3}\int \sin^2(x)\,dx\right] \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{1}{4}\int \sin^2(x)\,dx \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{1}{4}\int \frac{1}{2}\left(1-\cos(2x)\right)\;dx \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{1}{8}\int \left(1-\cos(w)\right)\;\frac{1}{2}\,dw \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{1}{16}\left(w-\sin(w)\right) + C \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{1}{16}\left(2x-\sin(2x)\right) + C \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{1}{16}\left(2x-2\sin(x)\cos(x)\right) + C \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{1}{8}\left(x-sin(x)\cos(x)\right) + C \\
&= \frac{1}{8}\left(2\cos(x)\sin^2(x) + x - \sin(x)\cos(x)\right) + C
\end{aligned}$$

... which is what we got in the other solution for 3, allowing for a little rearranging. ■

Quiz #2. Monday, 25 June. *[12 minutes]*

1. Compute $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$. *[5]*

SOLUTION. *(Using a trigonometric substitution.)* We see a component that looks like $\sqrt{1-x^2}$, so will use the trigonometric substitution $x = \sin(\theta)$, so $dx = \cos(\theta)\,d\theta$. (Note that we then have $\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta)$.) To evaluate the resulting trigonometric integral, we will us the substitution $u = \cos(\theta)$, so $du = -\sin(\theta)\,d\theta$ and $(-1)\,du = \sin(\theta)\,d\theta$.

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^3(\theta)}{\sqrt{1-\sin^2(\theta)}}\cos(\theta)\,d\theta = \int \frac{\sin^3(\theta)\cos(\theta)}{\sqrt{\cos^2(\theta)}}\,d\theta = \int \frac{\sin^3(\theta)\cos(\theta)}{\cos(\theta)}\,d\theta \\
&= \int \sin^3(\theta)\,d\theta = \int \sin^(\theta)\sin(\theta)\,d\theta = \int \left(1-\cos^2(\theta)\right)\sin(\theta)\,d\theta \\
&= \int \left(1-u^2\right)(-1)\,du = \int \left(u^2-1\right)\,du = \frac{u^3}{3} - u + C \\
&= \frac{1}{3}\cos^3(\theta) - \cos(\theta) + C = \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 - \sqrt{1-x^2} + C \\
&= \frac{1}{3}\left(1-x^2\right)^{3/2} - \left(1-x^2\right)^{1/2} + C \quad \square
\end{aligned}$$

SOLUTION. *(Using a non-trigonometric substitution.)* We will use the substitution $w = 1 - x^2$ to simplify the integral. Then $dw = -2x\,dx$, so $\left(-\frac{1}{2}\right)dw = x\,dx$; note also that $x^2 = 1 - w$.

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int \frac{x^2}{\sqrt{1-x^2}}\,x\,dx = \int \frac{1-w}{\sqrt{w}}\left(-\frac{1}{2}\right)dw = -\frac{1}{2}\int \left(\frac{1}{\sqrt{w}} - \frac{w}{\sqrt{w}}\right)dw \\
&= -\frac{1}{2}\int \left(w^{-1/2} - w^{1/2}\right)dw = \frac{1}{2}\int \left(w^{1/2} - w^{-1/2}\right)dw \\
&= \frac{1}{2}\left(\frac{w^{3/2}}{3/2} - \frac{w^{1/2}}{1/2}\right) + C = \frac{w^{3/2}}{3} - w^{1/2} + C \\
&= \frac{1}{3}\left(1-x^2\right)^{3/2} - \left(1-x^2\right)^{1/2} + C \quad \square
\end{aligned}$$

Quiz #3. Wednesday, 20 June. *[12 minutes]*

1. Compute $\displaystyle\int \frac{12}{x^3+4x}\,dx$. *[5]*

Solution. First, observe that $x^3+4x = x\left(x^2+4\right)$; moreover, since $x^2+4 \geq 4 > 0$ for all x, x^2+4 has no roots and hence is an irreducible quadratic. It follows that

$$\frac{12}{x^3+4x} = \frac{12}{x\left(x^2+4\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

for some constants A, B, and C.

We need to determine A, B, and C. Since

$$\frac{12}{x\left(x^2+4\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{A\left(x^2+4\right)+(Bx+C)\,x}{x\left(x^2+4\right)} = \frac{(A+B)x^2+Cx+4A}{x\left(x^2+4\right)},$$

we must have $(A+B)x^2+Cx+4A = 12$, so that $A+B=0$, $C=0$, and $4A=12$. From the last of these we have that $A = \frac{12}{4} = 3$, and it then follows from the first that $B = -A = -3$. Thus $A=3$, $B=-3$, and $C=0$, and so:

$$\int \frac{12}{x^3+4x}\,dx = \int \left(\frac{3}{x} + \frac{-3x+0}{x^2+4}\right)dx = 3\int \frac{1}{x}\,dx - 3\int \frac{x}{x^2+4}\,dx$$

The former integral is easy, but we'll have to put a bit more effort into the latter.

To compute $\int \frac{x}{x^2+4}\,dx$, we will use the substitution $u = x^2+4$, so that $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Thus:

$$\int \frac{x}{x^2+4}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\ln(u) + C = \frac{1}{2}\ln\left(x^2+4\right) + C$$

Putting all the pieces together, we have:

$$\int \frac{12}{x^3+4x}\,dx = 3\int \frac{1}{x}\,dx - 3\int \frac{x}{x^2+4}\,dx = 3\ln(x) - \frac{3}{2}\ln\left(x^2+4\right) + C \quad \blacksquare$$

Quiz #4. Wednesday, 4 July. *[12 minutes]*

Do *one* (1) of the following three questions.

1. How big does n have to be to guarantee that the Right-Hand Rule sum for $\displaystyle\int_0^2 (x+1)\,dx$ is within $0.1 = \frac{1}{10}$ of the exact value of the integral? *[5]*
2. How big does n have to be to guarantee that the Trapezoid Rule sum for $\displaystyle\int_0^2 (x+1)\,dx$ is within $0.1 = \frac{1}{10}$ of the exact value of the integral? *[5]*
3. The game of trigball is played with a double-pointed "ball" that is 10π *cm* long*. The cross-sections perpendicular to the axis of symmetry (which runs from one pointy end to the other) are circles of radius $10\sin(x)$ *cm*, where x is the distance (in *cm*) that cross section is from one end of the ball. Find the volume of a trigball. *[5]*

SOLUTIONS. 1. As was worked out in class, if $|f'(x)| \le M$ for all $x \in [a,b]$, the Right-Hand Rule sum $\frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$ differs from the exact value of $\int_a^b f(x)\,dx$ by at most $\frac{M(b-a)^2}{n}$. In this case $a = 0$ and $b = 2$, while $f'(x) = \frac{d}{dx}(x+1) = 1$ means that $M = 1$ will do. It follows that the Right-Hand Rule sum for n in this case differs from the integral $\int_0^n (x+1)\,dx$ by at most $\frac{1(2-0)^2}{n} = \frac{4}{n}$. In order to have the sum be guaranteed to be within 0.1 of the integral, we therefore have to ensure that $\frac{4}{n} \le 0.1$, that is, that $40 = \frac{4}{0.1} \le n$. □

NOTE: Neither in 1, nor in 2, were you actually asked to compute the relevant sum.

2. As is noted in the textbook (see Theorem 8.6.1 in §8.6), if $|f''(x)| \le M$ for all x in the interval $[a,b]$, the Trapezoid Rule sum for n differs from the integral it approximates by at most $\frac{M(b-a)^3}{12n^2}$. In this case $a = 0$ and $b = 2$, while $f''(x) = \frac{d^2}{dx^2}(x+1) = \frac{d}{dx}1 = 0$, so $M = 0$ will do. This means that the Trapezoid Rule sum for any $n \ge 1$ will differ from the integral by at most $\frac{0(2-0)^3}{12n^2} = 0$. That is, the Trapezoid Rule sum will give the exact answer no matter what $n \ge 1$ is for this integral. (Why?) □

3. To compute the volume of a solid, we integrate the area function of the cross-sections of the solid. In this case, the area of the circular cross-section at x, for $0 \le x \le 10\pi$, is $\pi r^2 = \pi\left(10\sin(x)\right)^2 = 100\pi\sin^2(x)$ (in cm^2, since $r = 10\sin(x)$ is in cm). Now:

$$V = \int_0^{10\pi} 100\pi\sin^2(x)\,dx = 100\pi\int_0^{10\pi} \frac{1}{2}\left(1-\cos(2x)\right)\,dx \qquad \begin{array}{l}\text{Substitute } u = 2x \text{, so}\\ du = 2\,dx \text{ and } dx = \frac{1}{2}\,du,\\ \text{with } \begin{array}{ccc} x & 0 & 10\pi \\ u & 0 & 20\pi \end{array}.\end{array}$$

$$= \frac{100\pi}{2}\int_0^{20\pi} \left(1-\cos(u)\right)\frac{1}{2}\,du = \frac{100\pi}{4}\int_0^{20\pi}\left(1-\cos(u)\right)\,du = 25\pi\left(u - \sin(u)\right)\Big|_0^{20\pi}$$

$$= 25\pi\left(20\pi - \sin(20\pi)\right) - 25\pi\left(0 - \sin(0)\right) = 25\pi\left(20\pi - 0\right) - 25\pi(0-0) = 500\pi$$

Thus the volume of a trigball is 500π cm^3. ■

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* The points are sharp. Please be careful when playing with a trigball.

Quiz #5. Wednesday, 11 July. *[10 minutes]*

1. Find the arc-length of $y = \dfrac{2}{3}x^{3/2}$ for $0 \leq x \leq 3$. *[5]*

SOLUTION. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2}\right) = \dfrac{2}{3} \cdot \dfrac{3}{2}x^{1/2} = x^{1/2}$. We will plug this into the integration formula for arc-length. In computing the resulting integral we will use the substitution $u = x+1$, so $du = dx$ and $\begin{matrix} x & 0 & 3 \\ u & 1 & 4 \end{matrix}$.

$$\text{Arc-length} = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 \sqrt{1 + \left(x^{1/2}\right)^2}\, dx = \int_0^3 \sqrt{1+x}\, dx = \int_1^4 \sqrt{u}\, du$$
$$= \int_1^4 u^{1/2}\, du = \left.\frac{u^{3/2}}{3/2}\right|_1^4 = \frac{2}{3} \cdot 4^{3/2} - \frac{2}{3} \cdot 1^{3/2} = \frac{2}{3} \cdot 8 - \frac{2}{3} = \frac{2}{3} \cdot 7 = \frac{14}{3} \quad \Box$$

Quiz #6. Monday, 16 July. *[15 minutes]*

Find the limit of each of the following sequences, if it exists. If the limit does not exist, give an informal explanation for why it doesn't.

1. $a_n = (-1)^n$ *[1]* 2. $b_n = \dfrac{n}{n^2+1}$ *[1]* 3. $c_n = \arctan(n)$ *[1]* 4. $d_n = \dfrac{n!}{2^n}$ *[2]*

SOLUTIONS. 1. $\lim_{n\to\infty} (-1)^n$ does not exist. If you start with $n = 0$, the sequence is $1, -1, 1, -1, 1, -1, \dots$ This fails to have a limit because it is not true that it gets close to one, and only one, real number. $\Box$

2. A little algebra goes a long way here:

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n^2+1}{n}} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} = 0$$

since $n + \dfrac{1}{n} \to \infty + 0 = \infty$ as $n \to \infty$. $\Box$

3. $\lim_{n\to\infty} \arctan(n) = \lim_{x\to\infty} \arctan(x) = \dfrac{\pi}{2}$ since $y = \arctan(x)$ has a horizontal asyptote of $y = \dfrac{\pi}{2}$ as $x \to \infty$. $\Box$

4. This limit fails to exist; in fact, $\lim_{n\to\infty} \dfrac{n!}{2^n} = \infty$. The first five terms of the sequence are $a_0 = \frac{0!}{2^0} = \frac{1}{1} = 1$, $a_1 = \frac{1!}{2^1} = \frac{1}{2}$, $a_2 = \frac{2!}{2^2} = \frac{2}{4} = \frac{1}{2}$, $a_3 = \frac{3!}{2^3} = \frac{6}{8} = \frac{3}{4}$, and $a_4 = \frac{4!}{2^4} = \frac{24}{16} = \frac{3}{2}$. After this point, $a_n > 1$ because if $a_{n-1} > 1$ and $n \geq 5$, then $a_n = a_{n-1} \cdot \dfrac{n}{2} > 1 \cdot \dfrac{5}{2} = \dfrac{5}{2} > 1$. Moreover, this same comparison hows that $a_n = a_{n-1} \cdot \dfrac{n}{2} > \dfrac{n}{2}$. Since $\lim_{n\to\infty} \dfrac{n}{2} = \infty$, we must then have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n!}{2^n} = \infty$. $\blacksquare$

Quiz #7. Wednesday, 18 July. *[20 minutes]*

Determine whether each of the following series converges or not.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1}}{e^{n+1}}$ *[1]* 2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ *[1]* 3. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{\sqrt{n^2+1}}$ *[1.5]* 4. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n)}{n\ln\left(2^n+1\right)}$ *[1.5]*

SOLUTIONS. 1. Observe that $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1}}{e^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2e}\left(\frac{2}{e}\right)^n$ is a geometric series with first term $a = \dfrac{1}{2e}$ and common ratio $r = \dfrac{2}{e}$. Since $2 < e \approx 2.718$, the common ratio $|r| = \dfrac{2}{e} < 1$, so the series converges. $\square$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ is a series where each term is a rational function in n. Considered as a polynomial in n, the numerator 1 has degree 0, while the denominator n^2-1 has degree 2. Since $p = 2-0 = 2 > 1$ for this series, the p-Test implies that the series converges. $\square$

3. Note that

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\sqrt{\frac{1}{n^2}\left(n^2+1\right)}}$$
$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{1+0}} = 1$$

because $\dfrac{1}{n^2} \to 0$ as $n \to \infty$. Since $\displaystyle\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1 \neq 0$, it follows by the Divergence Test that the series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{\sqrt{n^2+1}}$ diverges. $\square$

4. Whenever $n \geq 1$, we have

$$0 \leq \left|\frac{\cos(n)}{n\ln\left(2^n+1\right)}\right| \leq \frac{1}{n\ln\left(2^n\right)} \leq \frac{1}{n \cdot n\ln(2)} = \frac{1}{\ln(2)} \cdot \frac{1}{n^2},$$

so the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{n\ln\left(2^n+1\right)}\right|$ converges by comparison with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(2)} \cdot \frac{1}{n^2}$, which converges by the p-Test since it has $p = 2-0 = 2 > 1$. Thus the series $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n)}{n\ln\left(2^n+1\right)}$ is absolutely convergent, which means it's convergent. ■

Quiz #8. Monday, 23 July. *[20 minutes]*

Determine whether each of the following series converges or not.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+1}{5^n+2}$ *[1]* 2. $\displaystyle\sum_{n=1}^{\infty} \left[\ln\left(e^{-1/n}\right)\right]^n$ *[2]* 3. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{7}{n}\right)^n$ *[2]*

SOLUTIONS. 1. Looking at the dominant terms in $\dfrac{4^n+1}{5^n+2}$, it's a pretty good bet that the series should behave similarly to $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{5^n}$. As

$$\lim_{n\to\infty} \frac{\frac{4^n+1}{5^n+2}}{\frac{4^n}{5^n}} = \lim_{n\to\infty} \frac{4^n+1}{5^n+2} \cdot \frac{5^n}{4^n} = \lim_{n\to\infty} \frac{4^n+1}{5^n+2} \cdot \frac{\frac{1}{4^n}}{\frac{1}{5^n}} = \lim_{n\to\infty} \frac{1+\frac{1}{4^n}}{1+\frac{2}{5^n}} = \frac{1+0}{1+0} = 1,$$

it follows by the Limit Comparison Test that $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+1}{5^n+2}$ converges or diverges exactly as $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{5^n}$ does. Since $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n$ is a geometric series with first term $a = 1$ and common ratio $r = \dfrac{4}{5}$, and $|r| = \dfrac{4}{5} < 1$, it converges. Thus $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+1}{5^n+2}$ also converges. □

2. $\displaystyle\sum_{n=1}^{\infty} \left[\ln\left(e^{-1/n}\right)\right]^n = \sum_{n=1}^{\infty} \left[\frac{-1}{n}\ln(e)\right]^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^n}$ is obviously an alternating series. The underlying sequence of positive terms is decreasing, as $n+1 > n$ implies that $(n+1)^{n+1} > (n+1)^n > n^n$, which implies that $\dfrac{1}{(n+1)^{n+1}} < \dfrac{1}{n^n}$. Moreover, since $n^n \to \infty$ (very fast!) as $n \to \infty$, we have that $\displaystyle\lim_{n\to\infty} \frac{1}{n^n} = 0$. It now follows by the Alternating Series Test that $\displaystyle\sum_{n=1}^{\infty} \left[\ln\left(e^{-1/n}\right)\right]^n$ converges. (Alternatively, it's not hard to show that it converges absolutely, by an argument similar to that used in the solution to question 3 below, and hence converges.) □

3. Note that for all $n \geq 41^*$, we have $41^n < n^n$, so $0 < \left(\dfrac{7}{n}\right)^n < \left(\dfrac{7}{41}\right)^n$. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{7}{41}\right)^n$ is a geometric series with first term $a = \dfrac{7^2}{41^2}$ and common ratio $r = \dfrac{7}{41}$, which converges because $|r| = \dfrac{7}{41} < 1$. It now follows by the (Basic) Comparison Test that $\displaystyle\sum_{n=2}^{\infty} \left(\frac{7}{n}\right)^n$ converges as well. ■

* I used 41 because it's my favourite integer. Any real number greater than 7 would do just as well in this argument.

Quiz #9. Wednesday, 25 July. *[20 minutes]*

Find the radius and interval of convergence of each of the following power series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ *[1.5]* 2. $\displaystyle\sum_{n=0}^{\infty} \frac{5^{n+1}}{2^n} x^n$ *[1.5]* 3. $\displaystyle\sum_{n=0}^{\infty} \frac{n+3}{2n+1} x^n$ *[2]*

SOLUTIONS. 1. We will use the Root Test to find the radius of convergence. Since

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{x^n}{n^n}\right|^{1/n} = \lim_{n\to\infty} \left[\left(\frac{|x|}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{|x|}{n} = 0 < 1$$

for all $x \in \mathbb{R}$, it follows by the Root Test that $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ converges for all x, and hence the radius of convergence of this series is ∞ and its interval of convergence is $(-\infty, \infty)$. $\square$

2. We will use the fact that $\displaystyle\sum_{n=0}^{\infty} \frac{5^{n+1}}{2^n} x^n$ is a geometric series with common ratio $r = \dfrac{5x}{2}$. This means that it converges exactly when $|r| = \left|\dfrac{5x}{2}\right| < 1$, *i.e.* when $|x| < \dfrac{2}{5}$. It follows that the radius of convergence is $\dfrac{2}{5}$ and the interval of convergence is $\left(-\dfrac{2}{5}, \dfrac{2}{5}\right)$. $\square$

3. We will use the Ratio Test to find the radius of convergence. Since

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{(n+1)+3}{2(n+1)+1} x^{n+1}}{\frac{n+3}{2n+1} x^n}\right| = \lim_{n\to\infty} \frac{(n+4)|x|^{n+1}}{2n+3} \cdot \frac{2n+1}{(n+3)|x|^n} \\
&= \lim_{n\to\infty} \frac{2n^2+9n+4}{2n^2+9n+9} |x| = |x| \lim_{n\to\infty} \frac{2n^2+9n+4}{2n^2+9n+9} \cdot \frac{1/n^2}{1/n^2} \\
&= |x| \lim_{n\to\infty} \frac{2+\frac{9}{n}+\frac{1}{n^2}}{2+\frac{9}{n}+\frac{9}{n^2}} = |x| \cdot \frac{2+0+0}{2+0+0} = |x|,
\end{aligned}$$

the Ratio Test tells us that the series converges for all x with $|x| < 1$ and diverges for all x with $|x| > 1$. Thus the radius of convergence of this series is 1.

To find the interval of convergence, we need to determine what happens at the end-points, $x = -1$ and $x = 1$. When $x = -1$ we have the series $\displaystyle\sum_{n=0}^{\infty} \frac{n+3}{2n+1}(-1)^n$, which fails the Divergence Test. Since

$$\lim_{n\to\infty} \left|\frac{n+3}{2n+1}(-1)^n\right| = \lim_{n\to\infty} \frac{n+3}{2n+1} = \lim_{n\to\infty} \frac{n+3}{2n+1} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{2+\frac{1}{n}} = \frac{1+0}{2+0} = \frac{1}{2} \neq 0,$$

it cannot be true that $\displaystyle\lim_{n\to\infty} \frac{n+3}{2n+1}(-1)^n 0$. Thus, by the Divergence Test, $\displaystyle\sum_{m=0}^{\infty} \frac{n+3}{2n+1}(-1)^n$ does not converge. Similarly, when $x = 1$, we have the series $\displaystyle\sum_{n=0}^{\infty} \frac{n+3}{2n+1} 1^n = \sum_{n=0}^{\infty} \frac{n+3}{2n+1}$, which also fails the Divergence Test by the limit computed above. It follows that the interval of convergence of the given power series is $(-1, 1)$. $\blacksquare$