Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Solutions to Quiz #4

Do all three of the following questions. Please show all your work and simplify your answers as much as is reasonably possible, which might not be much.

1. Find the derivative of $g(x) = x^{1/2}\arctan\left(x^{1/2}\right)$. *[1.5]*

SOLUTION. We'll use the Product, Chain, and Power Rules:

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left(x^{1/2}\arctan\left(x^{1/2}\right)\right) = \left[\frac{d}{dx}x^{1/2}\right]\arctan\left(x^{1/2}\right) + x^{1/2}\left[\frac{d}{dx}\arctan\left(x^{1/2}\right)\right] \\
&= \left[\frac{1}{2}x^{-1/2}\right]\arctan\left(x^{1/2}\right) + x^{1/2}\left[\left.\frac{d}{dt}\arctan(t)\right|_{t=x^{1/2}} \cdot \frac{d}{dx}x^{1/2}\right] \\
&= \frac{1}{2}x^{-1/2}\arctan\left(x^{1/2}\right) + x^{1/2}\left[\left.\frac{1}{1+t^2}\right|_{t=x^{1/2}} \cdot \frac{1}{2}x^{-1/2}\right] \\
&= \frac{1}{2}x^{-1/2}\arctan\left(x^{1/2}\right) + x^{1/2} \cdot \frac{1}{1+\left(x^{1/2}\right)^2} \cdot \frac{1}{2}x^{-1/2} \\
&= \frac{1}{2}x^{-1/2}\arctan\left(x^{1/2}\right) + \frac{1}{2}x^{-1/2}x^{1/2} \cdot \frac{1}{1+x} \\
&= \frac{1}{2}x^{-1/2}\left(\arctan\left(x^{1/2}\right) + \frac{x^{1/2}}{1+x}\right) \quad \text{or (probably better)} \\
&= \frac{1}{2}\left(x^{-1/2}\arctan\left(x^{1/2}\right) + \frac{1}{1+x}\right) \qquad \square
\end{aligned}$$

2. Find the inverse function of $f(x) = \dfrac{\ln(2x) - \ln(x)}{\ln(2x) + \ln(x)}$. *[1.5]*

Hint: The fact that $\ln(ab) = \ln(a) + \ln(b)$ may come in handy.

SOLUTION. Note that this is an algebra problem: no limits or derivatives need apply here! We will apply the technique used in class and make use of the hint early on:

$$\begin{aligned}
y = f^{-1}(x) &\iff x = f(y) = \frac{\ln(2y) - \ln(y)}{\ln(2y) + \ln(y)} = \frac{\ln(2) + \ln(y) - \ln(y)}{\ln(2) + \ln(y) + \ln(y)} = \frac{\ln(2)}{\ln(2) + 2\ln(y)} \\
&\iff (\ln(2) + 2\ln(y))\, x = \ln(2) \iff \ln(2)x + 2\ln(y)x = \ln(2) \\
&\iff 2x\ln(y) = \ln(2) - \ln(2)x \iff \ln(y) = \frac{\ln(2) - \ln(2)x}{2x} = \frac{\ln(2)}{2} \cdot \frac{1-x}{x} \\
&\iff y = e^{\ln(y)} = e^{\frac{\ln(2)}{2}\left(\frac{1-x}{x}\right)} = \left(\left(e^{\ln(2)}\right)^{1/2}\right)^{(1-x)/x} = \left(2^{1/2}\right)^{(1-x)/x} \\
&\iff f^{-1}(x) = y = \left(\sqrt{2}\right)^{(1-x)/x} \qquad \textit{Whew!} \qquad \square
\end{aligned}$$

3. Find the derivative of $f(x) = \ln(\sec(x) + \tan(x))$. *[2]*

Solution One. *Simplify after only.* Off we go, with the help of the Chain Rule:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\ln(\sec(x)+\tan(x)) = \left.\frac{d}{dt}\ln(t)\right|_{t=\sec(x)+\tan(x)} \cdot \frac{d}{dx}(\sec(x)+\tan(x)) \\
&= \left.\frac{1}{t}\right|_{t=\sec(x)+\tan(x)} \cdot \left(\frac{d}{dx}\sec(x) + \frac{d}{dx}\tan(x)\right) \\
&= \frac{1}{\sec(x)+\tan(x)} \cdot \left(\sec(x)\tan(x) + \sec^2(x)\right) \\
&= \frac{\sec(x)\left(\tan(x)+\sec(x)\right)}{\sec(x)+\tan(x)} = \frac{\sec(x)\left(\sec(x)+\tan(x)\right)}{\sec(x)+\tan(x)} \\
&= \sec(x) \quad \square
\end{aligned}$$

Solution Two. *Simplify before ... and after, too.* Off we go, with the help of the Chain Rule and the fact that $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\ln(\sec(x)+\tan(x)) = \frac{d}{dx}\ln\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right) \\
&= \frac{d}{dx}\ln\left(\frac{1+\sin(x)}{\cos(x)}\right) = \frac{d}{dx}\left(\ln(1+\sin(x)) - \ln(\cos(x))\right) \\
&= \frac{d}{dx}\ln(1+\sin(x)) - \frac{d}{dx}\ln(\cos(x)) \\
&= \left.\frac{d}{dt}\ln(t)\right|_{t=1+\sin(x)} \cdot \frac{d}{dx}(1+\sin(x)) - \left.\frac{d}{du}\ln(u)\right|_{u=\cos(t)} \cdot \frac{d}{dt}\cos(x) \\
&= \left.\frac{1}{t}\right|_{t=1+\sin(x)} \cdot (0+\cos(x)) - \left.\frac{1}{u}\right|_{u=\cos(x)} \cdot (-\sin(x)) \\
&= \frac{1}{1+\sin(x)} \cdot \cos(x) - \frac{1}{\cos(x)} \cdot (-\sin(x)) = \frac{\cos(x)}{1+\sin(x)} + \frac{\sin(x)}{\cos(x)} \\
&= \frac{\cos(x)\cos(x) + (1+\sin(x))\sin(x)}{(1+\sin(x))\cos(x)} = \frac{\cos^2(x)+\sin(x)+\sin^2(x)}{(1+\sin(x))\cos(x)} \\
&= \frac{\cos^2(x)+\sin^2(x)+\sin(x)}{(1+\sin(x))\cos(x)} = \frac{1+\sin(x)}{(1+\sin(x))\cos(x)} = \frac{1}{\cos(x)} = \sec(x) \quad \square
\end{aligned}$$

Note: It's usually better to simplify, if you can, before taking the derivative, but there are exceptions ...