Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Quiz #3
Due just before midnight on Tuesday, 16 May.

Do all three of the following questions. Please show all your work and simplify your answers as much as is reasonably possible, which might not be much. You should not need to use the limit definition of the derivative.

1. Find the derivative of $h(x) = x^2\cos(x) + x^2e^x + 3x^2$. *[1.5]*

SOLUTION ONE. *Simplification before differentiation, as well as after.* We will factor out the x^2 each term has and then use the Product and Sum Rules:

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left(x^2\cos(x) + x^2e^x + 3x^2\right) = \frac{d}{dx}\left[x^2\left(\cos(x) + e^x + 3\right)\right] \\
&= \left[\frac{d}{dx}x^2\right]\cdot\left(\cos(x) + e^x + 3\right) + x^2\cdot\left[\frac{d}{dx}\left(\cos(x) + e^x + 3\right)\right] \\
&= 2x\cdot\left(\cos(x) + e^x + 3\right) + x^2\cdot\left(\left[\frac{d}{dx}\cos(x)\right] + \left[\frac{d}{dx}e^x\right] + \left[\frac{d}{dx}3\right]\right) \\
&= 2x\cdot\left(\cos(x) + e^x + 3\right) + x^2\cdot\left([-\sin(x)] + [e^x] + [0]\right) \\
&= 2x\cdot\left(\cos(x) + e^x + 3\right) + x^2\cdot\left(-\sin(x) + e^x\right) \\
&= x\left(2\cos(x) + 2e^x + 6 - x\sin(x) + xe^x\right) \\
&= x\left(2\cos(x) - x\sin(x) + (x+2)e^x + 6\right)
\end{aligned}$$

This is probably about as simplified as it gets, and perhaps more than is reasonable ... □

SOLUTION TWO. *No simplification until after differentiation.* We use the Sum and Product Rules and see what happens:

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left(x^2\cos(x) + x^2e^x + 3x^2\right) = \left[\frac{d}{dx}x^2\cos(x)\right] + \left[\frac{d}{dx}x^2e^x\right] + \left[\frac{d}{dx}3x^2\right] \\
&= \left[\left(\frac{d}{dx}x^2\right)\cos(x) + x^2\left(\frac{d}{dx}\cos(x)\right)\right] + \left[\left(\frac{d}{dx}x^2\right)e^x + x^2\left(\frac{d}{dx}e^x\right)\right] + \left[3\frac{d}{dx}x^2\right] \\
&= \left[2x\cos(x) + x^2\left(-\sin(x)\right)\right] + \left[2xe^x + x^2e^x\right] + \left[3\cdot 2x\right] \\
&= 2x\cos(x) - x^2\sin(x) + 2xe^x + x^2e^x + 6x \\
&= x\left(2\cos(x) - x\sin(x) + (x+2)e^x + 6\right)
\end{aligned}$$

About the same amount of work, in the end ... □

2. Find the derivative of $g(x) = \dfrac{x\sin(x)}{e^x}$. *[1.5]*

Solution. We will apply the Quotient and Product Rules:

$$\begin{aligned}
g'(x) = \frac{d}{dx}\left(\frac{x\sin(x)}{e^x}\right) &= \frac{\left[\frac{d}{dx}x\sin(x)\right]e^x + x\sin(x)\left[\frac{d}{dx}e^x\right]}{(e^x)^2} \\
&= \frac{\left[\left(\frac{d}{dx}x\right)\sin(x) + x\left(\frac{d}{dx}\sin(x)\right)\right]e^x + x\sin(x)e^x}{(e^x)^2} \\
&= \frac{\left[1\sin(x) + x\cos(x)\right]e^x + x\sin(x)e^x}{(e^x)^2} \\
&= \frac{\left(\sin(x) + x\cos(x) + x\sin(x)\right)e^x}{(e^x)^2} \\
&= \frac{\sin(x) + x\cos(x) + x\sin(x)}{e^x} \\
&= \frac{x\cos(x) + (x+1)\sin(x)}{e^x} \quad \text{or} \\
&= \frac{\sin(x) + x\left(\cos(x) + \sin(x)\right)}{e^x}, \text{ or just leave it be. } \square
\end{aligned}$$

3. Find the derivative of $f(x) = e^{3x}$ without using the Chain Rule. *[2]*

Solution. The trick here is to expand e^{3x} as a product and then apply – surprise! – the Product Rule:

$$\begin{aligned}
f''(x) &= \frac{d}{dx}e^{3x} = \frac{d}{dx}e^{x+x+x} = \frac{d}{dx}\left(e^x e^x e^x\right) = \left[\frac{d}{dx}\left(e^x e^x\right)\right]e^x + \left(e^x e^x\right)\left[\frac{d}{dx}e^x\right] \\
&= \left[\left(\frac{d}{dx}e^x\right)e^x + e^x\left(\frac{d}{dx}e^x\right)\right]e^x + \left(e^x e^x\right)\left[e^x\right] \\
&= \left[e^x e^x + e^x e^x\right]e^x + e^x e^x e^x \\
&= e^x e^x e^x + e^x e^x e^x + e^x e^x e^x \\
&= 3e^x e^x e^x = 3e^{x+x+x} = 3e^{3x} \quad \square
\end{aligned}$$

Note: This is much quicker if the Chain Rule is allowed.