**Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals**
TRENT UNIVERSITY, Summer 2023 (S61)

**Solutions to Quiz #2**

Do both of the following questions.

**1.** Compute $\displaystyle\lim_{x \to \infty} \frac{\left(x^2-1\right)\sin\left(\frac{\pi x}{2}\right)}{x^3+x^2-x-1}$ using what you've learned about limits. *[2.5]*

SOLUTION. We first use a little algebra to simplify the function we're taking the limit of:

$$\lim_{x \to \infty} \frac{\left(x^2-1\right)\sin\left(\frac{\pi x}{2}\right)}{x^3+x^2-x-1} = \lim_{x \to \infty} \frac{\left(x^2-1\right)\sin\left(\frac{\pi x}{2}\right)}{\left(x^2-1\right)(x+1)} = \lim_{x \to \infty} \frac{\sin\left(\frac{\pi x}{2}\right)}{x+1}$$

Observe that $-1 \leq \sin(t) \leq 1$ for all $t$ and that $x+1>0$ as $x \to \infty$. It follows that as $x \to \infty$,

$$\frac{-1}{x+1} \leq \frac{\sin\left(\frac{\pi x}{2}\right)}{x+1} \leq \frac{1}{x+1}\,.$$

As $x \to \infty$, $x+1 \to \infty$, too (it's one step ahead for every $x$), so

$$\lim_{x \to \infty} \frac{1}{x+1} = 0 \text{ and } \lim_{x \to \infty} \frac{-1}{x+1} = 0\,.$$

Hence, by the Squeeze Theorem, we have $\displaystyle\lim_{x \to \infty} \frac{\left(x^2-1\right)\sin\left(\frac{\pi x}{2}\right)}{x^3+x^2-x-1} = \lim_{x \to \infty} \frac{\sin\left(\frac{\pi x}{2}\right)}{x+1} = 0.$ ☐

**2.** Let $f(x)=(x-1)^2$. Compute $f'(x)$ using the limit definition of the derivative. *[2.5]*

SOLUTION. Here we go:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{\left((x+h)-1\right)^2-(x-1)^2}{h} \\
&= \lim_{h \to 0} \frac{\left((x+h)^2-2(x+h)+1\right)-\left(x^2-2x+1\right)}{h} \\
&= \lim_{h \to 0} \frac{\left(x^2+2xh+h^2-2x-2h+1\right)-\left(x^2-2x+1\right)}{h} \\
&= \lim_{h \to 0} \frac{x^2+2xh+h^2-2x-2h+1-x^2+2x-1}{h} \\
&= \lim_{h \to 0} \frac{2xh+h^2-2h}{h} = \lim_{h \to 0} \frac{h\,(2x+h-2)}{h} \\
&= \lim_{h \to 0} (2x+h-2) = 2x+0-2 = 2x-2 = 2(x-1) \quad \square
\end{aligned}$$