Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Solutions to Quiz #10
Integration

Please show all your work when answering the questions below. Do them by hand, please! Feel free to check your work using SageMath, though.

1. Work out $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$. *[2.5]*

SOLUTION. *i. One small step at a time.* We will work out the indefinite integral using consecutive small substitutions, a bit of algebra, and the Power Rule.

$$\begin{aligned}
\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx &= \int \frac{(e^x)^2}{\sqrt{e^x+1}}\,dx && \text{Substitute } u=e^x\text{, so } \tfrac{du}{dx}=e^x\text{, and thus } du=e^x\,dx.\\
&= \int \frac{u}{\sqrt{u+1}}\,du && \text{Now substitute } w=u+1\text{, so } \tfrac{dw}{du}=1\text{, and thus } dw=du \text{ and } u=w-1.\\
&= \int \frac{w-1}{\sqrt{w}}\,dw = \int\left(\sqrt{w}-\frac{1}{\sqrt{w}}\right)dw = \int\left(w^{1/2}-w^{-1/2}\right)dw\\
&= \frac{w^{3/2}}{3/2}-\frac{w^{1/2}}{1/2}+C = \frac{2}{3}w^{3/2}-2w^{1/2}+C && \text{Now substitute back \dots}\\
&= \frac{2}{3}(u+1)^{3/2}-2(u+1)^{1/2}+C && \text{\dots and again!}\\
&= \frac{2}{3}\left(e^x+1\right)^{3/2}-2\left(e^x+1\right)^{1/2}+C \quad \square
\end{aligned}$$

ii. "Whole hog." We'll do it all with just one very greedy substitution, some algebra, and the Power Rule.

$$\begin{aligned}
\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx &= \int \frac{(e^x)^2}{\sqrt{e^x+1}}\,dx && \text{Substitute } t=\sqrt{e^x+1}\text{, so } \tfrac{dt}{dx}=\tfrac{e^x}{2\sqrt{e^x+1}}\text{, and thus } 2\,dt=\tfrac{e^x}{\sqrt{e^x+1}}\,dx\text{. Also, } e^x=t^2-1.\\
&= \int \left(t^2-1\right)\cdot 2\,dt = 2\int\left(t^2-1\right)\,dt = 2\left(\frac{t^3}{3}-t\right)+C\\
\text{Substitute back.} \quad &= \frac{2}{3}t^3-2t+C = \frac{2}{3}\left(\sqrt{e^x+1}\right)^3-2\sqrt{e^x+1}+C
\end{aligned}$$

We leave it to you to check that this is the same as the answer above. $\square$

CHECK.

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In [3]: integral(e^(2*x)/sqrt(e^x + 1), x)
Out[3]: 2/3*(e^x + 1)^(3/2) - 2*sqrt(e^x + 1)
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Note that SageMath doesn't give the generic constant of integration.

2. Compute $\displaystyle\int_1^2 \frac{1}{\sqrt{x+1}+\sqrt{x-1}}\,dx$. *[2.5]*

Solution. We'll use a bit of algebra first to clear out the denominator, a couple of small substitutions after that, and the Power Rule.

$$\begin{aligned}
\int_1^2 \frac{1}{\sqrt{x+1}+\sqrt{x-1}}\,dx &= \int_1^2 \frac{1}{\sqrt{x+1}+\sqrt{x-1}}\cdot\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\,dx \\
&= \int_1^2 \frac{\sqrt{x+1}-\sqrt{x-1}}{\left(\sqrt{x+1}\right)^2-\left(\sqrt{x-1}\right)^2}\,dx \\
&= \int_1^2 \frac{\sqrt{x+1}-\sqrt{x-1}}{(x+1)-(x-1)}\,dx \\
&= \int_1^2 \frac{\sqrt{x+1}-\sqrt{x-1}}{2}\,dx \\
&= \frac{1}{2}\int_1^2 \left(\sqrt{x+1}-\sqrt{x-1}\right)\,dx
\end{aligned}$$

Now let $u = x+1$ and $w = x-1$, respectively, so $du = dw = dx$.

Changing limits:	x	u	w
	1	2	0
	2	3	1

$$\begin{aligned}
&= \frac{1}{2}\int_1^2 \sqrt{x+1}\,dx - \frac{1}{2}\int_1^2 \sqrt{x-1}\,dx \\
&= \frac{1}{2}\int_2^3 \sqrt{u}\,du - \frac{1}{2}\int_0^1 \sqrt{w}\,dw \\
&= \frac{1}{2}\int_2^3 u^{1/2}\,du - \frac{1}{2}\int_0^1 w^{1/2}\,dw \\
&= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}\bigg|_2^3 - \frac{1}{2}\cdot\frac{w^{1/2}}{3/2}\bigg|_0^1 \\
&= \left[\frac{1}{3}\cdot 3^{3/2} - \frac{1}{3}\cdot 2^{3/2}\right] - \left[\frac{1}{3}\cdot 1^{3/2} - \frac{1}{3}\cdot 0^{3/2}\right] \\
&= \frac{3\sqrt{3}-2\sqrt{2}}{3} - \frac{1-0}{3} = \frac{3\sqrt{3}-2\sqrt{2}-1}{3} \approx 0.4559 \quad \square
\end{aligned}$$

Check.

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In [4]: integral(1/(sqrt(x+1) + sqrt(x-1)), x, 1, 2)

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Out[4]: sqrt(3) - 2/3*sqrt(2) - 1/3
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No idea what those messages are about ...