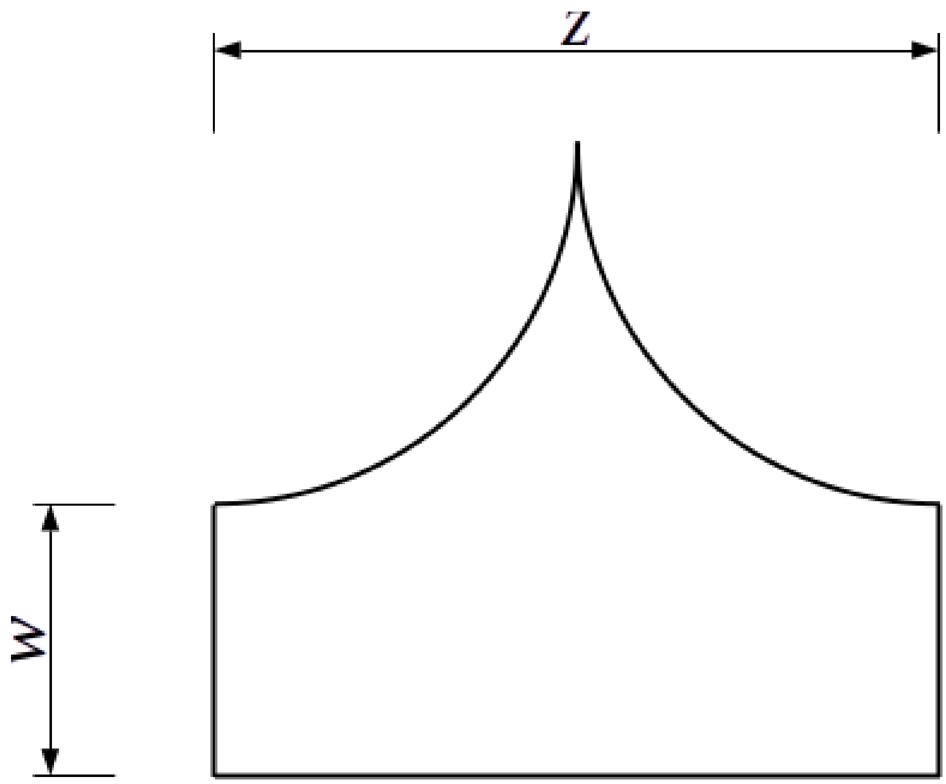
Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Solution to Assignment #4
Pointy Window



Architect Thom demands that a certain window be shaped as in the diagram above. (Start with a rectangle with base z and height $w + \frac{z}{2}$, and cut out the two quarter-circles of radius $\frac{z}{2}$ centered at the two upper corners of the rectangle.) The specific dimensions don't matter to Thom, just that the area be as large as possible, and the budget for glass for the project limits this window to an area of at most 4 m^2. On the other hand, the contractor wishes to minimize the perimeter of the window to keep the cost of the frame around it as low as possible.

1. What is the minimum possible perimeter of such a window, given that its area must be 4 m^2? *[10]*

   You may do this entirely by hand or use SageMath or similar software to help you out. In any case, show all your work!

SOLUTION. The solution below is done by hand and by SageMath in parallel. Interpreting the given problem and setting up the initial equations and expressions needed does have to be done by hand either way.

Considering the given diagram, it is pretty clear that the perimeter of the window, in terms of z and w, is:

$$\text{perimeter} = 2w + z + \frac{1}{2} \cdot 2\pi \cdot \frac{z}{2} = 2w + \frac{(2+\pi)z}{2}$$

(Recall that the perimeter of an entire circle of radius r is $2\pi r$.)

On the other hand, a rectangle with base z and height $w + \frac{z}{2}$ has area base $\cdot$ height $= z\left(w + \frac{z}{2}\right) = zw + \frac{z^2}{2}$. Two quarter-circles of radius $\frac{z}{2}$ have combined area $2 \cdot \frac{1}{4} \cdot \pi \left(\frac{z}{2}\right)^2 = \frac{\pi z^2}{8}$. (Recall that the area of an entire circle of radius r is πr^2.) Thus the area of the window in terms of z and w is:

$$\text{area} = zw + \frac{z^2}{2} - \frac{\pi z^2}{8} = zw + \frac{4z^2}{8} - \frac{\pi z^2}{8} = zw + \frac{(4-\pi)\, z^2}{8}$$

Since the area of the window is to be $4\ m^2$, we can solve for w in terms of z:

$$zw + \frac{(4-\pi)\, z^2}{8} = 4 \implies zw = 4 - \frac{(4-\pi)\, z^2}{8} \implies w = \frac{4}{z} - \frac{(4-\pi)\, z}{8}$$

This could also be done by using SageMath:

```
In [1]: var("w")
        var("z")
        A = z*(w + z/2) - (pi/2)*(z/2)^2 == 4
        solve(A,w)

Out[1]: [w == 1/8*((pi - 4)*z^2 + 32)/z]
```

Checking that this indeed the same – after some algebra! – is left to you, dear reader. We'll use the hand-generated expressions and functions when working by hand, and the SageMath-generated expressions and functions when working with SageMath from here on in.

Substituting this into the expression for the perimeter lets us write the perimeter as a function of z alone:

$$\begin{aligned} p(z) = \text{perimeter} = 2w + \frac{(2+\pi)z}{2} &= 2\left(\frac{4}{z} - \frac{(4-\pi)\, z}{8}\right) + \frac{(2+\pi)z}{2} \\ &= \frac{8}{z} - \frac{(4-\pi)\, z}{4} + \frac{(4+2\pi)z}{4} = \frac{8}{z} + \frac{3\pi z}{4} \end{aligned}$$

Note that this function, if one ignores where it came from, is defined for all $z \neq 0$. Since we are presumably sticking to physically possible windows, we will assume that $z > 0$. Note that there is an upper bound for z, too, coming from the area formula when $w = 4$. We won't bother with working it out here because it will be obvious later that that endpoint does not give a minimum for the perimeter.

To minimize the perimeter, we first look any critical point(s).

$$p'(z) = \frac{d}{dz}\left(\frac{8}{z} + \frac{3\pi z}{4}\right) = -\frac{8}{z^2} + \frac{3\pi}{4}$$

Then

$$p'(z) = 0 \iff -\frac{8}{z^2} + \frac{3\pi}{4} = 0 \iff \frac{8}{z^2} = \frac{3\pi}{4} \iff z^2 = \frac{32}{3\pi} \iff z = \pm\frac{4\sqrt{2}}{\sqrt{3\pi}},$$

and, since we areassumining $z > 0$, the only critical point we need to consider is $z = \frac{4\sqrt{2}}{\sqrt{3\pi}}$.

Doing all of the above calculations since the last time we used it in SageMath:

```
In [2]: Perim = function('Perim')(z)
        Perim(z) = z + pi*z/2 + 2*1/8*((pi - 4)*z^2 + 32)/z
        DPerim = diff(Perim,z)
        solve(DPerim == 0,z)

Out[2]: [z == -4*sqrt(2/3)/sqrt(pi), z == 4*sqrt(2/3)/sqrt(pi)]
```

We leave it to you to check that this is the same critical point.

Is this critical point a maximum, minimum, or neither? There are several ways to check; we will use the Second Derivative Test.

$$p''(z) = \frac{d}{dz}p'(z) = \frac{d}{dz}\left(-\frac{8}{z^2} + \frac{3\pi}{4}\right) = -\frac{-3 \cdot 8}{z^3} + 0 = \frac{24}{z^3}$$

Since $z^3 > 0$ when $z > 0$, $p''(z) > 0$ for all $z > 0$. It follows that our critical point is a local minimum and, since $p(z)$ is differentiable for all $z > 0$ and there is no other critical point with $z > 0$, it follows that our critical point is also a global minimum. (This is why we don't need to explicitly know what the maximum z possible from the area equation is.)

Of course, working out $p''(z)$ can also be done with SageMath:

```
In [3]: diff(DPerim,z)

Out[3]: z |--> -1/2*(pi - 4)/z + 1/2*((pi - 4)*z^2 + 32)/z^3
```

As is often the case, the way SageMath by default fails to simplify gives us something harder to work with. Once more we leave it to the reader to check whether the expressions obtained by hand and from SageMath are equivalent.

Having found that the minimum value of the perimeter occurs when $z = \frac{4\sqrt{2}}{\sqrt{3\pi}}$, it remains to plug this into the perimeter function and work out the minimum possible length of the perimeter:

$$p\left(\frac{4\sqrt{2}}{\sqrt{3\pi}}\right) = \frac{8}{\frac{4\sqrt{2}}{\sqrt{3\pi}}} + \frac{3\pi}{4} \cdot \frac{4\sqrt{2}}{\sqrt{3\pi}} = \sqrt{2} \cdot \sqrt{3\pi} + \sqrt{2} \cdot \sqrt{3\pi} = 2\sqrt{6\pi} \approx 8.6832150547$$

The decimal approximation was worked out on a scientific calculator.

Doing this with SageMath, copy-and-pasting the positive critical point it found into the perimeter function gives:

```
In [4]: Perim(4*sqrt(2/3)/sqrt(pi))

Out[4]: sqrt(2/3)*sqrt(pi)*((pi - 4)/pi + 3) + 2*sqrt(2/3)*sqrt(pi) + 4*sqrt
        (2/3)/sqrt(pi)

In [5]: N(sqrt(2/3)*sqrt(pi)*((pi - 4)/pi + 3) + 2*sqrt(2/3)*sqrt(pi) + 4*sqrt

Out[5]: 8.68321505469921
```

The lack of simplfication makes the symbolic answer a little hard to check against the hand-calculated answer, but looking at the decimal approximations gives a high degree of confidence that the answers are, in fact, the same.

Thus the minimum perimeter of a window of area 4 m^2 in Architect Thom's desired type of shape is $2\sqrt{6\pi} \approx 8.6832150547\ m$. $\square$