Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Assignment #2
Solving equations with and without SageMath

Before tackling this assignment, take a peek at the file `1110H-lab-20230510.pdf`, which you can find in the Labs folder in the Course Content section on Blackboard. Skimming and later referring to as necessary to Sections 1.8 and 4.8 of Gregory Bard's book *Sage for Undergraduates* (in the SageMath folder in the Course Content section on Blackboard) is probably a good idea. If you wish to use another general purpose mathematics application, such as Maple or Mathematica, you may, but you're on your own for learning to use it and getting help.

1. The Indian mathematician Bhaskara (1114-1185 A.D.), often referred to as Bhaskara II to distinguish him from an earlier mathematician named Bhaskara (*c.* 600-680 A.D.), posed the following problem in a book dedicated to his daughter Lilavati:

> *The square root of half the number of bees in a swarm has flown out upon a jasmine bush. Eight-ninths of the swarm has remained behind, and a female bee flies about a male who is buzzing inside a lotus flower; in the night, attracted by the flower's sweet odour, he went inside it, and now he is trapped! Tell me, most enchanting lady, the number of bees.*

The original text is actually in verse written in Sanskrit; the above is something of a loose translation. For those interested in the history of mathematics, Bhaskara developed a number of techniques that anticipated portions of both differential and integral calculus.

a. Express the information given in the problem as an equation. *[1]*
b. Solve the equation you obtained by hand. *[1]*
c. Solve the equation you obtained using SageMath. *[1]*
d. *Bonus!* What does this problem have to do with a Monty Python sketch? *[0.5]*

SOLUTIONS. **a.** Let x be the number of bees in the swarm. We are told that $\sqrt{x/2}$ of the bees fly away to a jasmine bush, $\frac{8}{9}x$ remain where they were, and 2 bees are in or near a lotus flower. Assuming all the bees are accounted for in the previous sentence, the number of bees, x, must satisfy the following equation: $x = \sqrt{\frac{x}{2}} + \frac{8}{9}x + 2$. □

b. Our strategy will be to isolate the square root, square both sides to get rid of the square root, simplify the resulting quadratic equation, and then solve it using the quadratic formula.

$$x = \sqrt{\frac{x}{2}} + \frac{8}{9}x + 2 \implies \sqrt{\frac{x}{2}} = x - \frac{8}{9}x - 2 = \frac{x}{9} - 2$$

$$\implies \frac{x}{2} = \left(\sqrt{\frac{x}{2}}\right)^2 = \left(\frac{x}{9} - 2\right)^2 = \frac{x^2}{81} - \frac{4}{9}x + 4$$

$$\implies 0 = \frac{x^2}{81} - \frac{4}{9}x - \frac{x}{2} + 4 = \frac{1}{81}x^2 - \frac{17}{18}x + 4$$

We can throw the quadratic formula at the equation $\frac{1}{81}x^2 - \frac{17}{18}x + 4 = 0$ right away, but the arithmetic is a little bit easier if we multiply through by 162, the least common multiple of the denominators 81 and 18, first, so that all the coefficients are integers.

$$\begin{aligned}
\frac{1}{81}x^2 - \frac{17}{18}x + 4 = 0 &\implies 2x^2 - 153x + 648 = 0 \\
&\implies x = \frac{-(-153) \pm \sqrt{(-153)^2 - 4 \cdot 2 \cdot 648}}{2 \cdot 2} \\
&\implies x = \frac{153 \pm \sqrt{23409 - 5184}}{4} = \frac{153 \pm \sqrt{18225}}{4} = \frac{153 \pm 135}{4} \\
&\implies x = \frac{288}{4} \text{ or } \frac{18}{4} = 72 \text{ or } \frac{9}{2}
\end{aligned}$$

Thus, unless we allow fractional bees, there are 72 bees in the swarm. □

c. Plugging the original equation from **a** into SageMath gives us something pretty useless:

```
[1]: solve(x == sqrt(x/2) + x*(8/9) + 2, x)

[1]: [x == 9/2*sqrt(2)*(2*sqrt(2) + sqrt(x))]
```

However, as soon as we get rid of the square root when doing things by hand in the solution to **b** above, we get a polynomial equation that SageMath handles nicely:

```
[2]: solve( x/2 == (x/9 - 2)^2, x)

[2]: [x == (9/2), x == 72]
```

It is, of course, still up to us to determine which of the possible answers is useful here. □

d. The English comedy troupe Monty Python wrote and performed the song *Eric the Half A Bee*, which may relate to the non-useful solution to Bhaskara's problem. You can find several instances of the song on YouTube, including one at:

`https://www.youtube.com/watch?v=ftomw87g61Y`

Be warned that part of it could be considered "not safe for work". □

2. The *hyperbolic functions* include:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The names of these function are usually pronounced something like "sinch", "kosh", "tanch", "co-seech", "seech", and "kotch", respectively. They turn out to be closely related to the natural exponential function (obviously) and the trigonometric functions;

the latter connections being more obvious when you look at their series expansions and expecially when you start looking at them as functions of a complex variable.

a. Use SageMath to compute $\lim_{x\to-\infty} \operatorname{sech}(x)$. *[1]*

b. Find a formula for the inverse function, arccsinh(x), of $\sinh(x)$ by hand. *[3]*

c. Find a formula for the inverse function of $\sinh(x)$ using SageMath. *[3]*

NOTE: Recall that a function $f(x)$ is the inverse of a function $g(x)$ if it undoes what $g(x)$ does, *i.e.* $f(g(x)) = x$. To put it another way, if $y = g(x)$, then $x = f(y)$, and *vice versa*.

Hint: $\operatorname{arcsinh}(x)$ is a natural logarithm of a quadratic expression. For fun and practice, try to figure out what its domain is once you're figured out what it is.

SOLUTIONS. **a.** Here we go:

```
[3]: lim(2/(e^x - e^(-x)),x=oo)
```

```
[3]: 0
```

Thus $\lim_{x\to-\infty} \operatorname{sech}(x) = 0$, which could be verified by hand fairly easily. □

b. Here we go, using the idea given in the note above. We will first isolate e^y, and then y.

$$\begin{aligned}
y = \operatorname{arcsinh}(x) &\iff x = \sinh(y) = \frac{e^y - e^{-y}}{2} \\
&\iff 2x = e^y - e^{-y} \\
&\iff 2xe^y = e^y e^y - e^{-y} e^y \\
&\iff 2xe^y = \left(e^y\right)^2 - 1 \\
&\iff \left(e^y\right)^2 - 2xe^y - 1 = 0 \\
&\qquad \text{This is a quadratic equation in } e^y \text{, to} \\
&\qquad \text{which we apply the quadratic formula:} \\
&\iff e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \\
&\iff e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1} \\
&\iff y = \ln\left(x \pm \sqrt{x^2+1}\right) = \begin{cases} \ln\left(x + \sqrt{x^2+1}\right) & \text{or} \\ \ln\left(x - \sqrt{x^2+1}\right) \end{cases}
\end{aligned}$$

This give us two alternative possibilities for a formula for $\operatorname{arcsinh}(x)$. The second one does not make sense: since $x^2 + 1 > x^2$ for all x, $\sqrt{x^2+1} > |x|$ for all x, so $x - \sqrt{x^2+1} < 0$ for all x, which means that $\ln\left(x - \sqrt{x^2+1}\right)$ is undefined for all x because $\ln(t)$ is only defined for $t > 0$. By the same token, though, the first alternative does make sense: since $x^2 + 1 > x^2$ for all x, $\sqrt{x^2+1} > |x|$ for all x, so $x + \sqrt{x^2+1} > 0$ for all x, which means $\ln\left(x - \sqrt{x^2+1}\right)$ is defined for all x. Thus $\operatorname{arcsinh}(x) = \ln\left(x - \sqrt{x^2+1}\right)$ for all x. □

c. It's pretty straightforward:

```
[4]: var("y")
     solve(x == (e^y - e^(-y))/2, y)

[4]: [y == log(x - sqrt(x^2 + 1)), y == log(x + sqrt(x^2 + 1))]
```

No massaging the equation required this time! We got the same two solutions we got by hand, though we still have to put in the effort to figure out which one makes sense. □