Trigonometric Integrals and Substitutions
A Brief Summary

0. A minimal set of trigonometric identities

- $\sin^2(x) + \cos^2(x) = 1$
  [Often used in the form $\cos^2(x) = 1 - \sin^2(x)$ or $\sin^2(x) = 1 - \cos^2(x)$.]
- $1 + \tan^2(x) = \sec^2(x)$
  [Sometimes used in the form $\sec^2(x) - 1 = \tan^2(x)$.]
- $\sin(2x) = 2\sin(x)\cos(x)$
- $$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= 2\cos^2(x) - 1\\ &= 1 - 2\sin^2(x)\end{aligned}$$
  [Sometimes used in the form $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ or $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$.]

It is also useful to keep in mind that:

- $\sin(x)$ and $\cos(x)$ are *periodic* with period 2π: for any real number x and any integer n, $\sin(x + 2n\pi) = \sin(x)$ and $\cos(x + 2n\pi) = \cos(x)$.
- $\sin(x)$ is an *odd* function, $\sin(-x) = -\sin(x)$ for all x, and $\cos(x)$ is an *even* function, $\cos(-x) = \cos(x)$ for all x.
- Phase shifts are fun: $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$, $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$, $\sin(x \pm \pi) = -\sin(x)$, and $\cos(x \pm \pi) = -\cos(x)$, for all x.

1. Some trigonometric integral reduction formulas

So long as $n \geq 2$, we have:

- $\displaystyle\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$
- $\displaystyle\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$
- $\displaystyle\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$
- $\displaystyle\int \sec^n(x)\,dx = \frac{1}{n-1}\tan(x)\sec^{n-2}(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$
- Just for fun – one usually looks this up as necessary – if we also have $k \geq 2$, then:

$$\begin{aligned}\int \sin^k(x)\cos^n(x)\,dx &= -\frac{\sin^{k-1}(x)\cos^{n+1}(x)}{k+n} + \frac{k-1}{k+n}\int \sin^{k-2}(x)\cos^n(x)\,dx\\ &= +\frac{\sin^{k+1}(x)\cos^{n-1}(x)}{k+n} + \frac{n-1}{k+n}\int \sin^k(x)\cos^{n-2}(x)\,dx\end{aligned}$$

For real obscurity, try to find or compute the corresponding formulas for integrands with mixed $\sec(x)$ and $\tan(x)$, not to mention the various reduction formulas involving $\csc(x)$ and/or $\cot(x)$.

2. Suggestions for trigonometric substitutions

A table of the basic forms:

If you see	*try substituting*	*so*	*and*
$\sqrt{1-x^2}$	$x=\sin(\theta)$	$dx=\cos(\theta)\,d\theta$	$\cos(\theta)=\sqrt{1-x^2}$
$\sqrt{1+x^2}$	$x=\tan(\theta)$	$dx=\sec^2(\theta)$	$\sec(\theta)=\sqrt{1+x^2}$
$\sqrt{x^2-1}$	$x=\sec(\theta)$	$dx=\sec(\theta)\tan(\theta)\,d\theta$	$\tan(\theta)=\sqrt{x^2-1}$

Here is a table of more general forms:

If you see	*try substituting*	*so*	*and*
$\sqrt{a^2-b^2x^2}$	$x=\frac{a}{b}\sin(\theta)$	$dx=\frac{a}{b}\cos(\theta)\,d\theta$	$\cos(\theta)=\frac{1}{a}\sqrt{a^2-b^2x^2}$
$\sqrt{a^2+b^2x^2}$	$x=\frac{a}{b}\tan(\theta)$	$dx=\frac{a}{b}\sec^2(\theta)\,d\theta$	$\sec(\theta)=\frac{1}{a}\sqrt{a^2+b^2x^2}$
$\sqrt{b^2x^2-a^2}$	$x=\frac{a}{b}\sec(\theta)$	$dx=\frac{a}{b}\sec(\theta)\tan(\theta)\,d\theta$	$\tan(\theta)=\frac{1}{a}\sqrt{b^2x^2-a^2}$

3. Handling arbitrary quadratics

How does one handle even more general situations with the square root of an arbitrary quadratic like $\sqrt{px^2+qx+r}$ (where $p\neq 0$) occurs in the integrand? In this case one "completes the square" on the quadratic,

$$px^2+qx+r = p\left[x^2+\frac{q}{p}x+\frac{r}{p}\right] = p\left[\left(x+\frac{q}{2p}\right)^2-\frac{q^2}{4p^2}+\frac{r}{p}\right]$$
$$= p\left(x+\frac{q}{2p}\right)^2+\left(r-\frac{q^2}{4p}\right),$$

and then uses a substitution like $u=x+\dfrac{q}{2p}$ to hopefully get a form like one of the "more general" ones above. If you get a form like $\sqrt{-b^2x^2-a^2}$ where what is inside the square root is always negative, you're out of luck unless you want to start doing calculus with complex numbers.*

4. Be alert to easier alternatives

Do not use the guidelines above without considering possible alternatives: a lot of integrals for which some trigonometric substitution works can also be handled, sometimes more easily, in other ways. For example, $\displaystyle\int x\sqrt{x^2-1}\,dx$ is probably most easily done with the basic substitution $u=x^2-1$.

* Take MATH 3770H in some later year, if you're interested. Complex analysis has some really fun results, such as Liouville's Theorem. Where there are plenty of non-constant differentiable functions with bounded output that are defined for all real numbers, such as $\sin(x)$, Liouville's Theorem asserts that every function that is defined and differentiable and bounded for all complex numbers is actually a constant function.