Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Summer 2021 (S62)
Solutions to Quiz #5
Wednesday, 21 July.

Do all of the following questions. Show all your reasoning in each solution. Please note that part marks are available in questions worth more than 0.5 points, so incomplete or incorrect solutions may still earn something.

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{3^n n}$ converges absolutely, converges conditionally, or diverges. *[1]*

Solution. This series converges absolutely. Consider the corresponding series of positive terms, namely

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n n}{3^n n} \right| = \sum_{n=0}^{\infty} \frac{n}{3^n n} = \sum_{n=0}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n .$$

This is a geometric series with common ratio $r = \frac{1}{3}$; as $|r| = \frac{1}{3} < 1$, it converges. As the corresponding series of positive terms converges, the given series converges absolutely. ■

2. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(n+2)(n+4)}$.

a. Verify that this series converges. *[1]*

b. Find the sum of this series. *[1]*

Solutions. **a.** We will use the basic form of the Comparison Test to check that the given series converges. Note that for all $n \geq 1$,

$$0 < \frac{2}{(n+2)(n+4)} = \frac{2}{n^2+6n+8} < \frac{2}{n^2} .$$

Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test (as it has $p = 2 > 1$), it follows by the basic Comparison Test that the given series converges. □

b. Observe that for all $n \geq 0$, we have

$$\begin{aligned} \frac{2}{(n+2)(n+4)} &= \frac{(n+4)-(n+2)}{(n+4)(n+2)} = \frac{(n+4)}{(n+4)(n+2)} - \frac{(n+2)}{(n+4)(n+2)} \\ &= \frac{1}{n+2} - \frac{1}{n+4} , \end{aligned}$$

so

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2}{(n+2)(n+4)} &= \sum_{n=0}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+4} \right) \\ &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \cdots \\ &= \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \cdots\right) + \left(\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} . \end{aligned}$$

Basically, the given series can be rewritten as the sum of two intertwined telescoping series. We get away with rearranging terms to separate out the two telescoping series, though we didn't absolutely need to do so, because the original series was a series of positive terms that converged, and hence converged absolutely. ■

3. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converges or diverges. *[2]*

SOLUTION. We will use the Limit Comparison Test (see the lecture *Series V*) to compare the given series to the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges by the p-Test since it has $p = \frac{3}{2} > 1$. Note that both series are positive after the first term. Computing the relevant limit,

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^2}}{\frac{\sqrt{n}}{n^2}} = \lim_{n\to\infty} \frac{\ln(n)}{n^2} \cdot \frac{n^2}{\sqrt{n}} = \lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = \lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} \begin{matrix} \to \infty \\ \to \infty \end{matrix}$$

It's an indeterminate form to which we can apply l'Hôpital's Rule:

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}x^{1/2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{2\sqrt{x}}{1}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}} \begin{matrix} \to 2 \\ \to \infty \end{matrix} = 0$$

Since we got a limit of 0 and the series whose terms we put in the denominator of the original limit converges, it follows by the Limit Comparison Test that the given series converges as well. ■

NOTE: There are several other ways to do each of these problems. For **3**, for example, we could have used the basic form of the Comparison Test (this would require showing that $\ln(n) < \sqrt{n}$ pas some point) or the Integral Test.

[Total = 5]