Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Summer 2021 (S62)
Quiz #2
Wednesday, 30 June.

Do all of the following questions. Show all your reasoning in each solution. Please note that part marks are available in questions worth more than 0.5 points, so incomplete or incorrect solutions may still earn something.

Compute each of the following integrals:

1. $\displaystyle\int \left(25+9x^2\right)^{3/2}\,dx$ *[2.5]*

SOLUTION. This will be a straightforward, but sadly tedious, apllication of the techniques discussed in lectures and the textbook for dealing with trigonometric substitutions and the resulting trigonometric integrals. The handout *Trigonometric Integrals and Substitutions: A Brief Summary* has pretty much everything we need.

The quadratic inside the power in the integrand is of the form $a^2+b^2x^2$, where $a=\sqrt{25}=5$ and $b=\sqrt{9}=3$ in our case. *Per* the advice in the handout, we should substitute $x=\frac{5}{3}\tan(\theta)$, so $dx=\frac{5}{3}\sec^2(\theta)\,\theta$. The aim is to eventually exploit the trigonometric identity $1+\tan^2(\theta)=\sec^2(\theta)$.

$$\begin{aligned}
\int \left(25+9x^2\right)^{3/2}\,dx &= \int \left(25+9\left(\frac{5}{3}\tan(\theta)\right)^2\right)^{3/2}\frac{5}{3}\sec^2(\theta)\,d\theta \\
&= \frac{5}{3}\int \left(25+9\cdot\frac{25}{9}\tan^2(\theta)\right)^{3/2}\sec^2(\theta)\,d\theta \\
&= \frac{5}{3}\int \left(25+25\tan^2(\theta)\right)^{3/2}\sec^2(\theta)\,d\theta \\
&= \frac{5}{3}\int 25^{3/2}\left(1+\tan^2(\theta)\right)^{3/2}\sec^2(\theta)\,d\theta \\
&= \frac{5}{3}\int 125\left(\sec^2(\theta)\right)^{3/2}\sec^2(\theta)\,d\theta \\
&= \frac{625}{3}\int \sec^3(\theta)\sec^2(\theta)\,d\theta = \frac{625}{3}\int \sec^5(\theta)\,d\theta
\end{aligned}$$

At this point, the most straightforward thing to do, tedious as it will be, is to use the secant integral reduction formula. This formula also happens to be on the handout mentioned above ...

$$\begin{aligned}
\int \left(25+9x^2\right)^{3/2}\,dx &= \frac{625}{3}\int \sec^5(\theta)\,d\theta \\
&= \frac{625}{3}\left[\frac{1}{5-1}\tan(\theta)\sec^{5-2}(\theta)+\frac{5-2}{5-1}\int \sec^{5-2}(\theta)\,d\theta\right] \\
&= \frac{625}{3}\left[\frac{1}{4}\tan(\theta)\sec^{3}(\theta)+\frac{3}{4}\int \sec^{3}(\theta)\,d\theta\right]
\end{aligned}$$

We apply the reduction formula again to the remaining integral.

$$\begin{aligned}
\int \left(25+9x^2\right)^{3/2}\, dx &= \frac{625}{3}\left[\frac{1}{4}\tan(\theta)\sec^3(\theta) + \frac{3}{4}\int \sec^3(\theta)\, d\theta\right] \\
&= \frac{625}{12}\tan(\theta)\sec^3(\theta) + \frac{625}{4}\int \sec^3(\theta)\, d\theta \\
&= \frac{625}{12}\tan(\theta)\sec^3(\theta) \\
&\quad + \frac{625}{4}\left[\frac{1}{3-1}\tan(\theta)\sec^{3-2}(\theta) + \frac{3-2}{3-1}\int \sec^{3-2}(\theta)\, d\theta\right] \\
&= \frac{625}{12}\tan(\theta)\sec^3(\theta) + \frac{625}{4}\left[\frac{1}{2}\tan(\theta)\sec(\theta) + \frac{1}{2}\int \sec(\theta)\, d\theta\right] \\
&= \frac{625}{12}\tan(\theta)\sec^3(\theta) + \frac{625}{8}\tan(\theta)\sec(\theta) + \frac{625}{8}\int \sec(\theta)\, d\theta \\
&= \frac{625}{12}\tan(\theta)\sec^3(\theta) + \frac{625}{8}\tan(\theta)\sec(\theta) \\
&\quad + \frac{625}{8}\ln\left(\sec(\theta)+\tan(\theta)\right) + C
\end{aligned}$$

It still remains to rewrite the antiderivative in terms of the original variable. Since we substituted $x = \frac{5}{3}\tan(\theta)$, it's pretty obvious that $\tan(\theta) = \frac{3}{5}x$; *per* the handout (if we're too lazy to do the algebra :-), $\sec(\theta) = \frac{1}{5}\sqrt{25+9x^2}$. Thus:

$$\begin{aligned}
\int \left(25+9x^2\right)^{3/2}\, dx &= \frac{625}{12}\tan(\theta)\sec^3(\theta) + \frac{625}{8}\tan(\theta)\sec(\theta) \\
&\quad + \frac{625}{8}\ln\left(\sec(\theta)+\tan(\theta)\right) + C \\
&= \frac{625}{12}\cdot\frac{3}{5}x\cdot\left(\frac{1}{5}\sqrt{25+9x^2}\right)^3 + \frac{625}{8}\cdot\frac{3}{5}x\cdot\frac{1}{5}\sqrt{25+9x^2} \\
&\quad + \frac{625}{8}\ln\left(\frac{1}{5}\sqrt{25+9x^2}+\frac{3}{5}x\right) + C \\
&= \frac{1}{4}x\left(25+9x^2\right)^{3/2} + \frac{75}{8}x\left(25+9x^2\right)^{1/2} \\
&\quad + \frac{625}{8}\ln\left(\frac{1}{5}\left(25+9x^2\right)^{1/2}+\frac{3}{5}x\right) + C
\end{aligned}$$

One could do more to "simplify" or rearrange this antiderivative, but it's probably not worth the bother unless one is given a good reason to do so.

As an alternative to doing a trigonometric substitution immediately, one can achieve the same effect in stages with multiple substitutions. First, substitute $x = \frac{5}{3}u$, so $dx = \frac{5}{3}\, du$; this will let's one kill off the 9 and then factor out the 25 in the integrand. Second, having reduced the quadratic expresssion to $1+u^2$ with the previous substitution and some algebra, one can substitute $u = \tan(\theta)$, and move on from there. ■

2. $\displaystyle\int_0^{\pi/8} \frac{1}{1-\tan^2(x)}\,dx$ *[2.5]* (Yes, that is a minus sign in the denominator.)

SOLUTION. The following solution is mostly an exercise in using trigonometric identities and algebra to put the integrand into a more useful form. Since none of the standard trigonometric identities seem helpful here, we resort to rewriting the integrand in terms of sine and cosine and seeing if we can do anything with the result:

$$\begin{aligned}
\int_0^{\pi/8} \frac{1}{1-\tan^2(x)}\,dx &= \int_0^{\pi/8} \frac{1}{1-\frac{\sin^2(x)}{\cos^2(x)}}\,dx \quad \text{using } \tan(x) = \frac{\sin(x)}{\cos(x)} \\
&= \int_0^{\pi/8} \frac{1}{\frac{\cos^2(x)}{\cos^2(x)} - \frac{\sin^2(x)}{\cos^2(x)}}\,dx \\
&= \int_0^{\pi/8} \frac{1}{\frac{\cos^2(x)-\sin^2(x)}{\cos^2(x)}}\,dx \\
&= \int_0^{\pi/8} \frac{\cos^2(x)}{\cos^2(x)-\sin^2(x)}\,dx
\end{aligned}$$

At this point we haul out the standard identities $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$, effectively rewriting the integrand in terms of $\cos(2x)$. Thus:

$$\begin{aligned}
\int_0^{\pi/8} \frac{1}{1-\tan^2(x)}\,dx &= \int_0^{\pi/8} \frac{\cos^2(x)}{\cos^2(x)-\sin^2(x)}\,dx \\
&= \int_0^{\pi/8} \frac{\frac{1}{2}+\frac{1}{2}\cos(2x)}{\cos(2x)}\,dx \\
&= \int_0^{\pi/8} \left[\frac{1}{2}\cdot\frac{1}{\cos(2x)} + \frac{1}{2}\cdot\frac{\cos(2x)}{\cos(2x)}\right]dx \\
&= \frac{1}{2}\int_0^{\pi/8} \sec(2x)\,dx + \frac{1}{2}\int_0^{\pi/8} 1\,dx
\end{aligned}$$

We now have it in a form that we can finish off. The second integral is – or at least darn well *should be* – easy, and the first we can put away with one of our hopefully memorized antiderivatives after a small substitution, namely $u = 2x$, so $du = 2\,dx$ and $dx = \frac{1}{2}\,du$. we will also change the limits as we go along: $\begin{matrix} x & 0 & \pi/8 \\ u & 0 & \pi/4 \end{matrix}$.

This gives us:

$$
\begin{aligned}
\int_0^{\pi/8} \frac{1}{1-\tan^2(x)}\, dx &= \frac{1}{2}\int_0^{\pi/8} \sec(2x)\, dx + \frac{1}{2}\int_0^{\pi/8} 1\, dx \\
&= \frac{1}{2}\int_0^{\pi/4} \sec(u)\, \frac{1}{2}\, du + \frac{1}{2}\cdot x\Big|_0^{\pi/8} \\
&= \frac{1}{4}\ln\left(\sec(u)+\tan(u)\right)\Big|_0^{\pi/4} + \left[\frac{1}{2}\cdot\frac{\pi}{8} - \frac{1}{2}\cdot 0\right] \\
&= \left[\frac{1}{4}\ln\left(\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)\right) - \frac{1}{4}\ln\left(\sec(0)+\tan(0)\right)\right] + \left[\frac{\pi}{16} - 0\right] \\
&= \left[\frac{1}{4}\ln\left(\sqrt{2}+1\right) - \frac{1}{4}\ln(1+0)\right] + \frac{\pi}{16} \\
&= \left[\frac{1}{4}\ln\left(\sqrt{2}+1\right) - \frac{1}{4}\cdot 0\right] + \frac{\pi}{16} = \left[\frac{1}{4}\ln\left(\sqrt{2}+1\right) - 0\right] + \frac{\pi}{16} \\
&= \frac{1}{4}\ln\left(\sqrt{2}+1\right) + \frac{\pi}{16}
\end{aligned}
$$

If you care about the decimal value, your calculator awaits the exercise! :-) ■

[Total = 5]