Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Summer 2021 (S62)
Solutions to Quiz #1
Wednesday, 23 June.

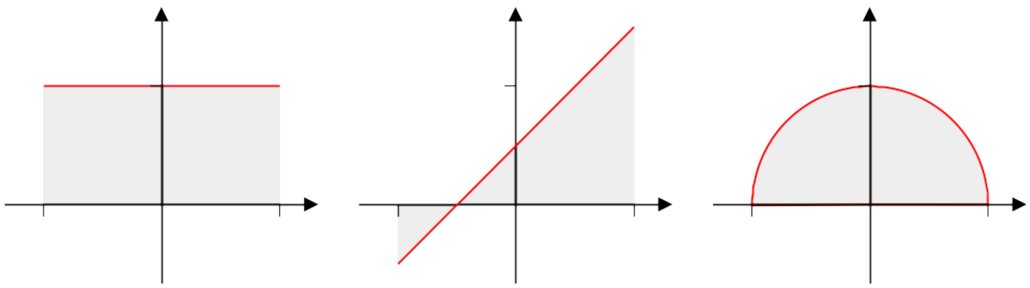
Do all of the following questions. Show all your reasoning in each solution. Please note that part marks are available in questions worth more than 0.5 points, so incomplete or incorrect solutions may still earn something.

1. Compute each of the following definite integrals *without doing any calculus*.

a. $\displaystyle\int_{-1}^{1} 1\,dx$ *[0.5]* **b.** $\displaystyle\int_{-1}^{1} \left(x+\frac{1}{2}\right)\,dx$ *[0.5]* **c.** $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$ *[0.5]*

Hint: All you need to know are what a definite integral represents and some geometry.

SOLUTIONS. Recall that a definite integral $\displaystyle\int_a^b f(x)\,dx$ represents the "weighted area" between the graph of $y=f(x)$ and the x-axis for $a \le x \le b$, where area above the x-axis is added and area below the x-axis is subtracted. Consider the three regions represented by the three given definite integrals:



a. In this case the region is rectangular, with width 2 and height 1, and is entirely above the x-axis, so $\displaystyle\int_{-1}^{1} 1\,dx = \text{width}\cdot\text{height} = 2\cdot 1 = 2$. □

b. In this case the region consists of two triangles, one with base and height $\frac{1}{2}$ below the x-axis and one with base and height $\frac{3}{2}$ above the x-axis. Since the area of a triangle with base b and height h is $\frac{1}{2}bh$, $\displaystyle\int_{-1}^{1}\left(x+\frac{1}{2}\right)dx = -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{2} = -\frac{1}{8}+\frac{9}{8} = \frac{8}{8} = 1$. □

c. In this case, the region is the upper half of a circle of radius 1, since $y=\sqrt{1-x^2}$ implies that $y^2 = 1-x^2$, so $x^2+y^2=1$. The area of an entire circle of radius r is πr^2, so it follows that $\displaystyle\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\cdot\pi 1^2 = \frac{\pi}{2}$. ■

2. Compute each of the following integrals.

a. $\displaystyle\int \frac{1+\cos(\ln(x))}{x}\,dx$ *[1]*

b. $\displaystyle\int_0^{\pi/4} x\arctan(x)\,dx$ *[1]*

c. $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$ *[1.5]*

Hints: **b.** $\tan(\pi/4)=1$ **c.** Go whole hog when substituting ...

SOLUTIONS. **a.** We will use the substitution $u=\ln(x)$, so $du=\frac{1}{x}\,dx$.

$$\int \frac{1+\cos(\ln(x))}{x}\,dx = \int (1+\cos(u))\,du = u+\sin(u)+C = \ln(x)+\sin(\ln(x))+C \quad \square$$

b. Since the integrand is a product of two dissimilar functions, we will start by doing integration by parts, with $u=\arctan(x)$ and $v'=x$, so $u'=\frac{1}{1+x^2}$ and $v=\frac{x^2}{2}$.

$$\begin{aligned}
\int_0^{\pi/4} x\arctan(x)\,dx &= \arctan(x)\cdot\frac{x^2}{2}\Big|_0^{\pi/4} - \int_0^{\pi/4}\frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx \\
&= \frac{x^2\arctan(x)}{2}\Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\frac{x^2}{1+x^2}\,dx \\
&\quad \text{Cheap algebraic trick time!} \\
&= \left[\frac{\left(\frac{\pi}{4}\right)^2\arctan\left(\frac{\pi}{4}\right)}{2} - \frac{0^2\arctan(0)}{2}\right] - \frac{1}{2}\int_0^{\pi/4}\frac{1+x^2-1}{1+x^2}\,dx \\
&= \left[\frac{\left(\frac{\pi}{4}\right)^2\cdot 1}{2} - 0\right] - \left[\frac{1}{2}\int_0^{\pi/4}\frac{1+x^2}{1+x^2}\,dx - \frac{1}{2}\int_0^{\pi/4}\frac{1}{1+x^2}\,dx\right] \\
&= \frac{\pi^2}{32} - \frac{1}{2}\int_0^{\pi/4} 1\,dx + \frac{1}{2}\int_0^{\pi/4}\frac{1}{1+x^2}\,dx \qquad \text{... and now take a peek at } u' \text{ above.} \\
&= \frac{\pi^2}{32} - \frac{1}{2}x\Big|_0^{\pi/4} + \arctan(x)\big|_0^{\pi/4} \\
&= \frac{\pi^2}{32} - \left[\frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\cdot 0\right] + \left[\arctan\left(\frac{\pi}{4}\right) - \arctan(0)\right] \\
&= \frac{\pi^2}{32} - \left[\frac{\pi}{8} - 0\right] + \left[\arctan\left(\frac{\pi}{4}\right) - 0\right] = \frac{\pi^2}{32} - \frac{\pi}{8} + \arctan\left(\frac{\pi}{4}\right)
\end{aligned}$$

The final answer is not pretty, but that's pretty common in real life. Note also that the hint was something of a red herring. $\square$

c. There are several ways to use substitution to solve this problem. The simplest way to proceed is probably the substitution $u = e^x$, followed by $w = u + 1$, but in this solution we will instead show off what happens when you try your hardest to do it in one go. Thus we will substitute "whole hog" with $u = \sqrt{e^x + 1}$. Rather than working out du directly from the substitution, which is a bit messy, we will first solve for x in terms of u:

$$u = \sqrt{e^x + 1} \implies u^2 = e^x + 1 \implies e^x = u^2 - 1 \implies x = \ln\left(u^2 - 1\right)$$

... and then get dx in terms of u and du:

$$dx = \frac{dx}{du}\,du = \frac{d}{du}\ln\left(u^2 - 1\right)\,du = \frac{1}{u^2 - 1}\cdot\frac{d}{du}\left(u^2 - 1\right)\,du = \frac{2u}{u^2 - 1}\,du$$

This trick of solving for the original variable in terms of the new variable is often helpful when doing complicated substitutions. Anyway, let's see what we get when we execute the substitution. Recall that we obtained $e^x = u^2 - 1$ above, so $e^{2x} = \left(e^x\right)^2 = \left(u^2 - 1\right)^2$. Now

$$\begin{aligned}
\int \frac{e^{2x}}{\sqrt{e^x + 1}}\,dx &= \int \frac{\left(e^x\right)^2}{\sqrt{e^x + 1}}\,dx = \int \frac{\left(u^2 - 1\right)^2}{u}\cdot\frac{2u}{u^2 - 1}\,du = \int 2\left(u^2 - 1\right)\,du \\
&= 2\left(\frac{u^3}{3} - u\right) + C = 2\left(\frac{\left(\sqrt{e^x + 1}\right)^3}{3} - \sqrt{e^x + 1}\right) + C \\
\text{or} \quad &= \frac{2}{3}\left(e^x + 1\right)^{3/2} - 2\left(e^x + 1\right)^{1/2} + C \\
\text{or} \quad &= 2\left(e^x + 1\right)^{1/2}\left(\frac{1}{3}\left(e^x + 1\right) - 1\right) + C = 2\left(e^x + 1\right)^{1/2}\left(\frac{e^x}{3} - \frac{2}{3}\right) + C\,,
\end{aligned}$$

among the many possible ways to write the final antiderivative.

Is this better than proceeding by way of simpler substitutions? Not significantly, if at all, but the way the algebra works out in this case – the way things cancel out when you make the substitution – is kind of cool. ■

[Total = 5]