Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Summer 2021 (S62)

Solutions to the Take-Home Final Examination
Released at noon on Wednesday, 28 July, 2021.
Due by noon on Saturday, 31 July, 2021.
Corrected on 2021-07-29.

INSTRUCTIONS

- You may consult your notes, handouts, and textbook from this course and any other math courses you have taken or are taking now. You may also use a calculator. However, you may not consult any other source, or give or receive any other aid, except for asking the instructor to clarify instructions or questions.
- Please submit an electronic copy of your solutions, preferably as a single pdf (a scan of handwritten solutions should be fine), via the Assignment module on Blackboard. If that doesn't work, please email your solutions to the intructor. *Show all your work!*
- Do all three (3) of Parts **X**, **Y**, and **Z**, and, if you wish, Part **B** as well.

Part X. Do both of **1** and **2**. *[40 = 2×20 each]*

1. Compute the integrals in any five (5) of **a** – **f**. *[20 = 5×4 each]*

a. $\displaystyle\int_0^{\pi/2} \sin(2x)\cos^2(x)\,dx$ **b.** $\displaystyle\int \frac{x+1}{x^3+x}\,dx$ **c.** $\displaystyle\int_1^e x\left(\ln(x)\right)^2\,dx$

d. $\displaystyle\int_0^{\pi/4} \tan^2(x)\sec^2(x)\,dx$ **e.** $\displaystyle\int \frac{\sqrt{1-x^2}}{\left(x^2-1\right)^2}\,dx$ **f.** $\displaystyle\int e^x\cosh(x)\,dx$

SOLUTIONS. **a.** We will use the trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$, followed by the substitution $u = \cos(x)$, so $du = -\sin(x)\,dx$ and $\sin(x)\,dx = (-1)\,du$, and change the limits as we go along: $\begin{matrix} x & 0 & \pi/2 \\ u & 1 & 0 \end{matrix}$. Here we go:

$$\begin{aligned}\int_0^{\pi/2} \sin(2x)\cos^2(x)\,dx &= \int_0^{\pi/2} 2\sin(x)\cos(x)\cos^2(x)\,dx = 2\int_0^{\pi/2}\cos^3(x)\sin(x)\,dx \\ &= 2\int_1^0 u^3(-1)\,du = 2\int_0^1 u^3\,du = 2\cdot\left.\frac{u^4}{4}\right|_0^1 = \left.\frac{u^4}{2}\right|_0^1 = \frac{1^4}{2}-\frac{0^4}{2} \\ &= \frac{1}{2}-0=\frac{1}{2} \qquad \square\end{aligned}$$

b. We will use partial fractions to decompose the integrand into digestible pieces. First, the denominator factors easily, $x^3+x = x\left(x^2+1\right)$, into a linear factor, x, and an irreducible quadratic factor, x^2+1. x^2+1 is irreducible because $x^2+1 \geq 1 > 0$ for all x, so it has no roots and hence cannot be factored into linear factors. (Alternatively, using the quadratic formula gives you roots that are not real numbers.)

Second, it follows from this that we can decompose the integrand in terms of partial fractions as $\dfrac{x+1}{x^3+x} = \dfrac{x+1}{x\left(x^2+1\right)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$, where A, B, and C are constants.

Third, we solve for the constants A, B, and C. Putting the right-hand side of the equation above over a common denominator tells us that:

$$\frac{x+1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A\left(x^2+1\right)+(Bx+C)\,x}{x\left(x^2+1\right)} = \frac{(A+B)x^2+Cx+A}{x\left(x^2+1\right)}$$

Thus $x+1=(A+B)x^2+Cx+A$, so we must have $A+B=0$, $C=1$ and $A=1$, from which it follows that $B=-A=-1$, and so $\dfrac{x+1}{x^3+x} = \dfrac{x+1}{x\left(x^2+1\right)} = \dfrac{1}{x} + \dfrac{-x+1}{x^2+1}$.

Fourth, we use the partial fraction decomposition of the integrand to break up the integral and compute it piecemeal:

$$\int \frac{x+1}{x^3+x}\,dx = \int \left(\frac{1}{x} + \frac{-x+1}{x^2+1}\right)dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

In the middle integral we will substitute $w=x^2+1$, so $dw=2x\,dx$ and $x\,dx=\frac{1}{2}\,du$.

$$= \ln(x) - \int \frac{1}{u}\cdot\frac{1}{2}\,du + \arctan(x) = \ln(x) - \frac{1}{2}\ln(u) + \arctan(x) + K$$

$$= \ln(x) - \frac{1}{2}\ln\left(x^2+1\right) + \arctan(x) + K$$

We used K for the constant of integration because we had already used C in a different way in a previous step. One could use the properties of logarithms to combine the logarithmic components in the final answer to get $\ln\left(\dfrac{x}{\sqrt{x^2+1}}\right) + \arctan(x) + K$, if one wished to. □

c. We will use integration by parts, with $u=(\ln(x))^2$ and $v'=x$, so $u'=2\ln(x)\cdot\dfrac{1}{x}$ and $v=\dfrac{x^2}{2}$, and see what we can do:

$$\int_1^e x\,(\ln(x))^2\,dx = \left.\frac{x^2\,(\ln(x))^2}{2}\right|_1^e - \int_1^e 2\ln(x)\cdot\frac{1}{x}\cdot\frac{x^2}{2}\,dx$$

$$= \left(\frac{e^2\,(\ln(e))^2}{2} - \frac{1^2\,(\ln(1))^2}{2}\right) - \int_1^e x\ln(x)\,dx$$

We use parts again: $s=\ln(x)$ and $t'=x$, so $s'=\frac{1}{x}$ and $t=\frac{x^2}{2}$.

$$= \left(\frac{e^2 1^2}{2} - \frac{1^2 0^2}{2}\right) - \left[\left.\frac{x^2\ln(x)}{2}\right|_1^e - \int_1^e \frac{1}{x}\cdot\frac{x^2}{2}\,dx\right]$$

$$= \left(\frac{e^2}{2} - 0\right) - \left[\left(\frac{e^2\ln(e)}{2} - \frac{1^2\ln(1)}{2}\right) - \frac{1}{2}\int_1^e x\,dx\right]$$

$$= \frac{e^2}{2} - \left[\left(\frac{e^2 1}{2} - \frac{1^2 0}{2}\right) - \frac{1}{2}\cdot\left.\frac{x^2}{2}\right|_1^e\right] = \frac{e^2}{2} - \left[\left(\frac{e^2}{2} - 0\right) - \left(\frac{e^2}{4} - \frac{1^2}{4}\right)\right]$$

$$= \frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{4} - \frac{1}{4} = \frac{e^2-1}{4} \qquad \square$$

d. We'll use the substitution $w = \tan(x)$, so $dw = \sec^2(x)\,dx$, and change the limits as we go along: $\begin{array}{ccc} x & 0 & \pi/4 \\ w & 0 & 1 \end{array}$

$$\int_0^{\pi/4} \tan^2(x)\sec^2(x)\,dx = \int_0^1 w^2\,dw = \left.\frac{w^3}{3}\right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} - 0 = \frac{1}{3} \qquad \square$$

e. First, observe that

$$\int \frac{\sqrt{1-x^2}}{\left(x^2-1\right)^2}\,dx = \int \frac{\sqrt{1-x^2}}{\left(1-x^2\right)^2}\,dx = \int \frac{1}{\left(1-x^2\right)^{3/2}}\,dx\,.$$

Second, we will use the trigonometric substitution $x = \sin(\theta)$, so $dx = \cos(\theta)\,d\theta$ and $\left(1-x^2\right)^{1/2} = \sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta)$.

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{\left(x^2-1\right)^2}\,dx &= \int \frac{1}{\left(1-x^2\right)^{3/2}}\,dx = \int \frac{1}{\left[\left(1-x^2\right)^{1/2}\right]^3}\,dx = \int \frac{1}{[\cos(\theta)]^3}\,\cos(\theta)\,d\theta \\
&= \int \frac{1}{\cos^2(\theta)}\,d\theta = \sec^2(\theta)\,d\theta = \tan(\theta) + C = \frac{\sin(\theta)}{\cos(\theta)} + C \\
&= \frac{x}{\sqrt{1-x^2}} + C \qquad \square
\end{aligned}$$

f. We'll use the fact that, by definition, $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$:

$$\begin{aligned}
\int e^x \cosh(x)\,dx &= \int e^x \cdot \frac{e^x + e^{-x}}{2}\,dx = \int \frac{e^x e^x + e^x e^{-x}}{2}\,dx = \frac{1}{2}\int \left(e^{2x} + e^0\right)\,dx \\
&= \frac{1}{2}\int \left(\left(e^x\right)^2 + 1\right)\,dx = \frac{1}{2}\int \left(e^x\right)^2\,dx + \frac{1}{2}\int 1\,dx \\
&\qquad \text{Now substitute } u = e^x, \text{ so } du = e^x\,dx, \text{ in the first integral.} \\
&= \frac{1}{2}\int u\,du + \frac{x}{2} = \frac{1}{2}\cdot\frac{u^2}{2} + \frac{x}{2} + C = \frac{\left(e^x\right)^2}{4} + \frac{x}{2} + C \\
&= \frac{e^{2x}}{4} + \frac{x}{2} + C \qquad \blacksquare
\end{aligned}$$

2. Determine whether the series converges in any five (5) of **a** – **f**. *[20 = 5×4 each]*

a. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ **b.** $\displaystyle\sum_{n=0}^{\infty} \frac{41^n}{n(n+1)}$ **c.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!\cdot 2^n}{(2n)!}$

d. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2-1}{\left(n^2+1\right)^2}$ **e.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!+2^n}$ **f.** $\displaystyle\sum_{n=100}^{\infty} \frac{\sin(n\pi)+\cos(n\pi)}{\ln\left(e^n\right)}$

SOLUTIONS. **a.** This is a task for the Integral Test. Note that the function $f(x) = \dfrac{1}{x\sqrt{\ln(x)}}$ is positive and decreasing for all $x \geq 3$, and that $f(n)$ is the nth term of the series for all $n \geq 3$.

$$\int_3^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,dx = \lim_{a\to\infty} \int_3^{a} \frac{1}{x\sqrt{\ln(x)}}\,dx$$

Now substitute $u = \ln(x)$, so $du = \frac{1}{x}\,dx$, and change the limits: $\begin{array}{ccc} x & 3 & a \\ u & \ln(3) & \ln(a) \end{array}$

$$= \lim_{a\to\infty} \int_{\ln(3)}^{\ln(a)} \frac{1}{\sqrt{u}}\,du = \lim_{a\to\infty} \int_{\ln(3)}^{\ln(a)} u^{-1/2}\,du$$

$$= \lim_{a\to\infty} \left.\frac{u^{1/2}}{1/2}\right|_{\ln(3)}^{\ln(a)} \lim_{a\to\infty} 2\sqrt{u}\Big|_{\ln(3)}^{\ln(a)}$$

$$= \lim_{a\to\infty} \left(2\sqrt{\ln(a)} - 2\sqrt{\ln(3)}\right) = \infty$$

since $\ln(a) \to \infty$ as $a \to \infty$ and $\sqrt{\ln(a)} \to \infty$ as $\ln(a) \to \infty$.

By the Integral Test, since the improper integral $\displaystyle\int_3^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,dx$ diverges, so does the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$. □

b. *Divergence Test.* With a bit of help from l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{41^n}{n(n+1)} = \lim_{x\to\infty} \frac{41^x}{x(x+1)} \begin{array}{l}\to\infty\\ \to\infty\end{array} = \lim_{x\to\infty} \frac{\frac{d}{dx}41^x}{\frac{d}{dx}\left(x^2+x\right)} = \lim_{x\to\infty} \frac{\ln(41)41^x}{2x+1} \begin{array}{l}\to\infty\\ \to\infty\end{array}$$

$$= \lim_{x\to\infty} \frac{\ln(41)\frac{d}{dx}41^x}{\frac{d}{dx}(2x+1)} = \lim_{x\to\infty} \frac{\left(\ln(41)\right)^2 41^x}{2} \begin{array}{l}\to\infty\\ \to 2\end{array} = \infty \neq 0$$

Since $\displaystyle\lim_{n\to\infty} \frac{41^n}{n(n+1)} \neq 0$, the Divergence Test tells us that the given series diverges. □

b. *Ratio Test.* We plug the terms of this series into the Ratio Test and see what happens. Notice that all the terms are positive.

$$\lim_{n\to\infty}\left|\frac{\frac{41^{n+1}}{(n+1)(n+1+1)}}{\frac{41^n}{n(n+1)}}\right| = \lim_{n\to\infty}\frac{41^{n+1}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{41^n} = \lim_{n\to\infty}\frac{41n}{n+2}$$
$$= \lim_{n\to\infty}\frac{41n}{n+2}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{41}{1+\frac{2}{n}} = \frac{41}{1+0} = 41 > 1$$

Since the limit asked for by the Ratio Test is greater than 1, the series $\displaystyle\sum_{n=0}^{\infty}\frac{41^n}{n(n+1)}$ diverges. $\square$

c. This one is tailor-made for the Ratio Test, as the terms are built using multiplication and division only. Notice that all the terms are positive.

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)!\cdot 2^{n+1}}{(2(n+1))!}}{\frac{n!\cdot 2^n}{(2n)!}}\right| = \lim_{n\to\infty}\frac{(n+1)!\cdot 2^{n+1}}{(2(n+1))!}\cdot\frac{(2n)!}{n!\cdot 2^n} = \lim_{n\to\infty}\frac{(n+1)\cdot 2}{(2n+2)(2n+1)}$$
$$= \lim_{n\to\infty}\frac{1}{2n+1}\begin{matrix}\to 1\\ \to\infty\end{matrix} = 0 < 1$$

Since the limit asked for by the Ratio Test is less than 1, the given series converges. $\square$

NOTE: The series in **c** can also be shown to be convergent with the Comparison Test – try comparing it to the series $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2}$ – but there is a fair bit of algebra to wade through ...

d. We will use the Limit Comparison Test to compare the given series, $\displaystyle\sum_{n=0}^{\infty}\frac{n^2-1}{(n^2+1)^2}$ (note that all the terms past $n=1$ are positive), to the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$.

$$\lim_{n\to\infty}\frac{\frac{n^2-1}{(n^2+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2-1}{(n^2+1)^2}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{n^4-n^2}{n^4+2n^2+1} = \lim_{n\to\infty}\frac{n^4-n^2}{n^4+2n^2+1}\cdot\frac{\frac{1}{n^4}}{\frac{1}{n^4}}$$
$$= \lim_{n\to\infty}\frac{1-\frac{1}{n^2}}{1+\frac{2}{n^2}+\frac{1}{n^4}} = \frac{1-0}{1+0+0} = 1 \quad\text{as } \frac{1}{n^2}\to 0 \text{ and } \frac{1}{n^4}\to 0 \text{ as } n\to\infty.$$

Since the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-Test since it has $p=2>1$, it follows by the Limit Comparison test that the original series converges as well. $\square$

e. Observe that for all $n \geq 0$, $0 \leq \dfrac{3^n}{n!+2^n} \leq \dfrac{3^n}{n!}$ since decreasing the denominator increases the fraction. The Basic Comparison Test will therefore allow us to conclude that the given series converges as long as we can verify that $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges. We verify that it does so using the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \lim_{n\to\infty} \frac{3}{n+1} \begin{matrix} \to 3 \\ \to \infty \end{matrix} = 0 < 1$$

It follows by the Ratio Test that $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges, and hence, by the Comparison Test, that $\sum_{n=0}^{\infty} \frac{3^n}{n! + 2^n}$ converges as well. □

f. $\sin(x) = 0$ whenever x is an integer multiple of π, so $\sin(n\pi) = 0$ for all n. On the other hand, $\cos(x)$ is ± 1 when x is an integer multiple of π, with $\cos(n\pi) = 1$ when n is even and $\cos(n\pi) = -1$ when n is odd, so $\cos(n\pi) = (-1)^n$ for all n. Also, since $\ln(x)$ and e^x are each other's inverse functions, $\ln(e^n) = n$ for all $n \geq 1$. It follows that the given series,

$$\sum_{n=100}^{\infty} \frac{\sin(n\pi) + \cos(n\pi)}{\ln(e^n)} = \sum_{n=100}^{\infty} \frac{0 + (-1)^n}{n} \sum_{n=100}^{\infty} \frac{(-1)^n}{n},$$

is the alternating harmonic series in disguise. This converges by the Alternating Series Test since the series satisfies the necessary conditions:

i. The series alternates sign because $\frac{1}{n} > 0$ for all $n > 0$ and $(-1)^n$ alternates sign.

ii. For all $n > 0$, $\left| \frac{(-1)+1}{n+1} \right| = \frac{1}{n+1} < \frac{1}{n} = \left| \frac{(-1)^n}{n} \right|$, *i.e.* the terms of the series decrease in absolute value.

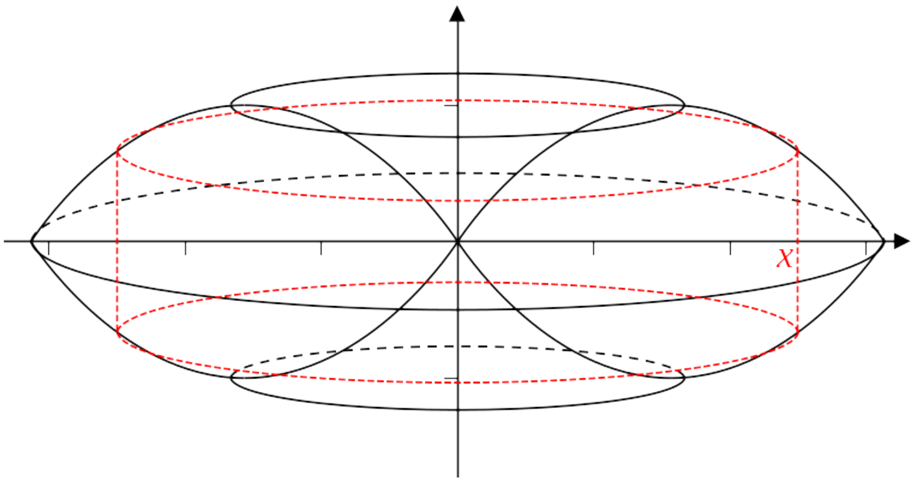
iii. $\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0.$

It follows that the given series converges. ■

Part Y. Do any three (3) of **3** – **6**. *[30 = 3×10 each]*

3. Sketch the solid obtained by revolving the region below $y = \sin(x)$ and above $y = -\sin(x)$, for $0 \le x \le \pi$, about the y-axis, and find its volume. *[10]*

SOLUTION. Here is sketch of the solid, with the cylindrical shell at x drawn in:

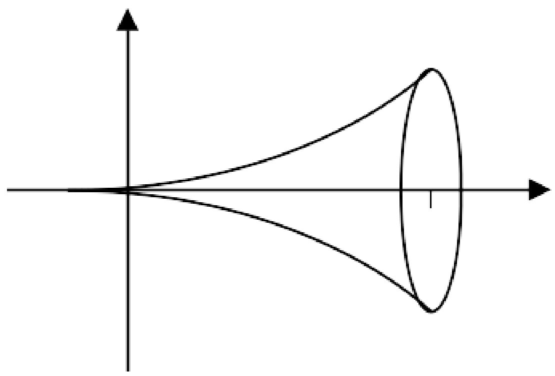


As one might guess from the inclusion of a cylindrical shell in the sketch, we will use the method of cylindrical shells to compute the volume of the solid. Since we rotated the region about the y-axis, the shells are perpendicular to the x-axis, and so we use x as our variable. The cylindrical shell at x has radius $r = x - 0 = x$ and height $h = \sin(x) - (-\sin(x)) = 2\sin(x)$, and hence has area $2\pi rh = 2\pi x \cdot 2\sin(x) = 4\pi x \sin(x)$. Since the original region had $0 \le x \le \pi$, the volume of this solid of revolution is:

$$
\begin{aligned}
\text{Volume} &= \int_0^\pi 2\pi rh\,dx = \int_0^\pi 4\pi x \sin(x)\,dx = 4\pi \int_0^\pi x\sin(x)\,dx \\
&\quad \text{We'll use integration by parts, with } u = x \text{ and } v' = \sin(x), \\
&\quad \text{so } u' = 1 \text{ and } v = -\cos(x). \\
&= 4\pi \left(-x\cos(x)\big|_0^\pi - \int_0^\pi 1\,(-\cos(x))\,dx \right) \\
&= 4\pi \left([(-\pi\cos(\pi)) - (-0\cos(0))] + \int_0^\pi \cos(x)\,dx \right) \\
&= 4\pi \left([-\pi(-1) - 0] + \sin(x)\big|_0^\pi \right) \\
&= 4\pi \left(\pi + [\sin(\pi) - \sin(0)] \right) \\
&= 4\pi \left(\pi + [0 - 0] \right) = 4\pi^2 \qquad \blacksquare
\end{aligned}
$$

4. Find the area of the surface obtained by revolving the curve $y = \frac{x^3}{3}$, for $0 \le x \le 1$, about the x-axis. *[10]*

SOLUTION. Although it's not needed, here is a sketch of the surface:



Since $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3}{3}\right) = \frac{3x^2}{3} = x^2$, the chunk of arc-length at x is given by

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + (x^2)^2}\,dx = \sqrt{1 + x^4}\,dx\,.$$

This chunk of arc-length is revolved around the x-axis through a circle with radius $r = y - 0 = \frac{x^3}{3}$ which has circumference $2\pi r = 2\pi\frac{x^3}{3}$. It follows that the area of the surface obtained by revolving the curve $y = \frac{x^3}{3}$, for $0 \le x \le 1$, about the x-axis is:

$$\text{Surface Area} = \int_0^1 2\pi r\,ds = \int_0^1 2\pi\frac{x^3}{3}\sqrt{1 + x^4}\,dx = \frac{2\pi}{3}\int_0^1 x^3\sqrt{1 + x^4}\,dx$$

We substitute $w = 1 + x^4$, so $dw = 4x^3\,dx$ and $x^3\,dx = \frac{1}{4}\,dw$,

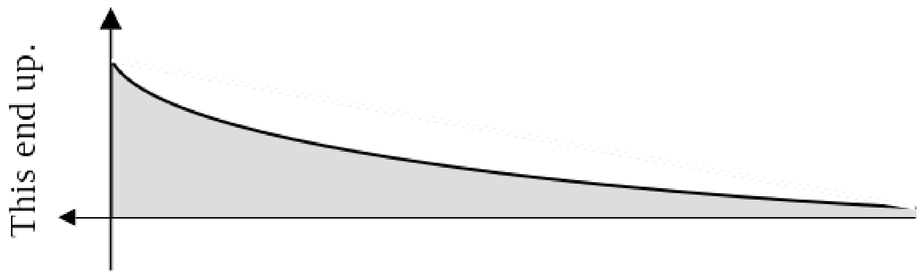
and change the limits accordingly: $\begin{array}{ccc} x & 0 & 1 \\ w & 1 & 2 \end{array}$

$$= \frac{2\pi}{3}\int_1^2 \sqrt{w}\,\frac{1}{4}\,dw = \frac{2\pi}{3}\cdot\frac{1}{4}\int_1^2 w^{1/2}\,dw = \frac{\pi}{6}\cdot\left.\frac{w^{3/2}}{3/2}\right|_1^2 = \left.\frac{\pi}{6}\cdot\frac{2}{3}w^{3/2}\right|_1^2$$

$$= \left.\frac{\pi}{9}w^{3/2}\right|_1^2 = \frac{\pi}{9}2^{3/2} - \frac{\pi}{9}1^{3/2} = \frac{\pi}{9}\left(2\sqrt{2} - 1\right) \qquad \blacksquare$$

5. Find the area of the region below $y = 0$ and above $y = \ln(x)$, where $0 < x \leq 1$. *[10]*

SOLUTION. Although it's not needed, here is a sketch of the region:



The area integral is easy enough to set up:

$$\text{Area} = \int_0^1 (0 - \ln(x))\ dx = -\int_0^1 \ln(x)\, dx = \int_1^0 \ln(x)\, dx$$

Since $\ln(x)$ has a vertical asymptote at 0, though, this is an improper integral and should be computed accordingly:

$$\text{Area} = \int_1^0 \ln(x)\, dx = \lim_{a \to 0^+} \int_1^a \ln(x)\, dx$$

We now use parts, with $u = \ln(x)$ and $v' = 1$, so $u' = \frac{1}{x}$ and $v = x$.

$$= \lim_{a \to 0^+} \left[x\ln(x) - \int \frac{1}{x} \cdot x\, dx \right]_1^a = \lim_{a \to 0^+} \left[x\ln(x) - \int 1\, dx \right]_1^a = \lim_{a \to 0^+} \left[x\ln(x) - x \right]_1^a$$

$$= \lim_{a \to 0^+} \left[(a\ln(a) - a) - (1 \cdot \ln(1) - 1) \right] = \lim_{a \to 0^+} \left[a\ln(a) - a - 1 \cdot 0 + 1 \right]$$

$$= \left[\lim_{a \to 0^+} a\ln(a) \right] - \left[\lim_{a \to 0^+} a \right] - 0 + 1 = \left[\lim_{a \to 0^+} \frac{\ln(a) \to -\infty}{\frac{1}{a} \quad \to +\infty} \right] - 0 - 0 + 1$$

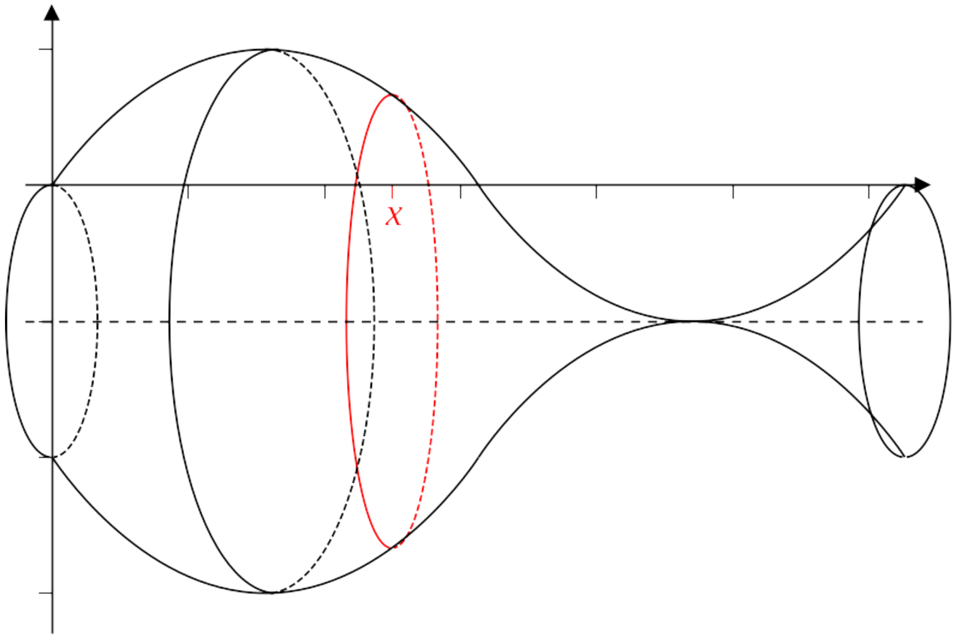
We'll use l'Hôpital's Rule to compute the indeterminate limit.

$$= \left[\lim_{a \to 0^+} \frac{\frac{d}{da} \ln(a)}{\frac{d}{da}\left(\frac{1}{a}\right)} \right] + 1 = \left[\lim_{a \to 0^+} \frac{\frac{1}{a}}{\frac{-1}{a^2}} \right] + 1 = \left[\lim_{a \to 0^+} \frac{1}{a} \cdot \frac{a^2}{-1} \right] + 1 = \left[\lim_{a \to 0^+} (-a) \right] + 1$$

$$= -0 + 1 = 1 \qquad \blacksquare$$

6. Sketch the solid obtained by revolving the region below $y = \sin(x)$ and above $y = -1$, for $0 \leq x \leq 2\pi$, about the ~~x-axis~~ line $y = -1$, and find its volume. *[10]*

SOLUTION. Here is sketch of the solid, with the disk at x drawn in:



As one might guess from the inclusion of a disk in the sketch, we will use the disk/washer method to compute the volume of the solid. Note that since the axis of revolution is part of the region's border, the cross-sections are disks rather than washers. Since we rotated the region about the horizontal line $y = -1$, the cross-sectional disks are perpendicular to the x-axis, and so we use x as our variable. The disk at x has radius $r = \sin(x) - (-1) = \sin(x) + 1$ and hence area $\pi r^2 = \pi\left(\sin(x) + 1\right)^2$. Since the original region had $0 \leq x \leq 2\pi$, the volume of this solid of revolution is:

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \pi r^2\, dx = \int_0^{2\pi} \pi\left(\sin(x)+1\right)^2\, dx = \pi \int_0^{2\pi} \left(\sin^2(x) + 2\sin(x) + 1\right)\, dx \\
&= \pi \int_0^{2\pi} \sin^2(x)\, dx + \pi \int_0^{2\pi} 2\sin(x)\, dx + \pi \int_0^{2\pi} 1\, dx \\
&= \pi \left[-\frac{1}{2}\sin^{2-1}(x)\cos(x) + \frac{2-1}{2}\int \sin^{2-2}(x)\, dx\right]_0^{2\pi} + 2\pi\left(-\cos(x)\right)\Big|_0^{2\pi} + \pi x\Big|_0^{2\pi} \\
&= \pi \left[-\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}\int 1\, dx\right]_0^{2\pi} + 2\pi\left[-\cos(2\pi) - (-\cos(0))\right] + \left[\pi \cdot 2\pi - \pi \cdot 0\right] \\
&= \pi \left[-\frac{1}{2}\sin(x)\cos(x) + \frac{x}{2}\right]_0^{2\pi} + 2\pi\left[-1 - (-1)\right] + \left[2\pi^2 - 0\right] \\
&= \pi \left[\left(-\frac{1}{2}\sin(2\pi)\cos(2\pi) + \frac{2\pi}{2}\right) - \left(-\frac{1}{2}\sin(0)\cos(0) + \frac{0}{2}\right)\right] + 2\pi \cdot 0 + 2\pi^2 \\
&= \pi \left[\left(-\frac{1}{2} \cdot 0 \cdot 1 + \pi\right) - \left(-\frac{1}{2} \cdot 0 \cdot 1 + 0\right)\right] + 0 + 2\pi^2 \\
&= \pi\left[(0+\pi) - (0+0)\right] + 2\pi^2 = \pi^2 + 2\pi^2 = 3\pi^2 \qquad \blacksquare
\end{aligned}$$

Part Z. Do any three (3) of **7** – **10**. *[30 = 3×10 each]*

7. Determine the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2^n} x^n$. *[10]*

SOLUTION. We will attack this using the Ratio Test, as usual.*

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{2^{n+1}} x^{n+1}}{\frac{n!}{2^n} x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n! x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x}{2} \right|$$
$$= \lim_{n\to\infty} \frac{(n+1)|x|}{2} = \begin{cases} \infty & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since $\infty > 1$ and $0 < 1$, it follows by the Ratio Test that this power series, which is centred at 0, diverges when $x \neq 0$ and converges only when $x = 0$. That is, its radius of convergence is $R = 0$ and its interval of convergence is $[0, 0] = \{0\}$. ■

8. Consider the function $f(x) = \sin(x) + \sinh(x)$.

a. Use Taylor's formula to find the Taylor series centred at 0 of $f(x)$. *[4]*

b. Determine the radius and interval of convergence of this Taylor series. *[3]*

c. Find the Taylor series centred at 0 of $f(x)$ without using Taylor's formula. *[3]*

SOLUTIONS. **a.** Recall that $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, $\dfrac{d}{dx}\sinh(x) = \cosh(x) = \dfrac{e^x + e^{-x}}{2}$, and $\dfrac{d}{dx}\cosh(x) = \sinh(x)$, in contrast with $\dfrac{d}{dx}\sin(x) = \cos(x)$ and $\dfrac{d}{dx} cos(x) = \sin(x)$. Note that $\sin(0) = \sinh(0) = 0$ and $\cos(0) = \cosh(0) = 1$. We will generate the usual table for $f^{(n)}(0)$ and attempt to spot a pattern.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x) + \sinh(x)$	0
1	$\cos(x) + \cosh(x)$	2
2	$-\sin(x) + \sinh(x)$	0
3	$-\cos(x) + \cosh(x)$	0
4	$\sin(x) + \sinh(x)$	0
5	$\cos(x) + \cosh(x)$	2
6	$-\sin(x) + \sinh(x)$	0
7	$-\cos(x) + \cosh(x)$	0
8	$\sin(x) + \sinh(x)$	0
9	$\cos(x) + \cosh(x)$	2
$\vdots$	$\vdots$	$\vdots$

* Another reason is that the main alternative, the Root Test, would have us dealing with $(n!)^{1/n}$, which would be a pain even with with the help of Stirling's formula, which states that for large n, $n!$ is approximately $\sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$.

The pattern here is not hard to discern: when $n = 4k+1$ for some $k \geq 0$, then $f^{(n)}(0) = 2$, and otherwise $f^{(n)}(0) = 0$. Plugging this into Taylor's formula, it follows that the Taylor series centred at 0 of $f(x) = \sin(x) + \sinh(x)$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{\infty} \frac{f^{(4k+1)}(0)}{(4k+1)!} x^{4k+1} = \sum_{k=0}^{\infty} \frac{2}{(4k+1)!} x^{4k+1}$$
$$= 2x + \frac{2}{120} x^5 + \frac{2}{362880} x^9 + \cdots \qquad \square$$

b. To find the radius and interval of convergence of the series obtained above, we once again begin with the Ratio Test.

$$\lim_{k\to\infty} \left| \frac{\frac{2}{(4(k+1)+1)!} x^{4(k+1)+1}}{\frac{2}{(4k+1)!} x^{4k+1}} \right| = \lim_{k\to\infty} \left| \frac{2x^{4k+5}}{(4k+5)!} \cdot \frac{(4k+1)!}{2x^{4k+1}} \right|$$
$$= \lim_{k\to\infty} \frac{x^4}{(4k+5)(4k+4)(4k+3)(4k+2)} \frac{\to x^4}{\to \infty} = 0$$

Since $0 < 1$, it follows by the Ratio Test that the Taylor series centred at 0 of $f(x) = \sin(x) + \sinh(x)$ converges for all x, so it has radius of convergence $R = \infty$ and interval of convergence $(-\infty, \infty)$. $\square$

c. Recall that the Taylor series centred at 0 of $\sin(x)$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and that the Taylor series centred at 0 of e^x is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Substituting $-x$ in for x in the latter series tells us that the Taylor series centred at 0 of e^{-x} is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$. It follows that the Taylor series centred at 0 of $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ is:

$$\frac{\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) - \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right)}{2} = \frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)}{2}$$
$$= \frac{1}{2}\left(2x + 2\frac{x^3}{6} + 2\frac{x^5}{120} + \cdots\right) = x + \frac{x^3}{6} + \frac{x^5}{120} + \cdots$$
$$= \sum n = 0^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

This, in turn, means that the Taylor series centred at 0 of $f(x) = \sin(x) + \sinh(x)$ is:

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] + \left[\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\right] = \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right] + \left[x + \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right]$$
$$= 2x + \frac{2x^5}{120} + \frac{2x^9}{362880} + \cdots = \sum_{k=0}^{\infty} \frac{2x^{4k+1}}{(4k+1)!} \qquad \blacksquare$$

9. Find the Taylor series centred at π of $f(x) = \sin(x)$ and determine its radius and interval of convergence. *[10]*

SOLUTION. Recall that $\dfrac{d}{dx}\sin(x) = \cos(x)$ and $\dfrac{d}{dx}cos(x) = \sin(x)$, and that $\sin(\pi) = 0$ and $\cos(\pi) = -1$. We will generate the usual table for $f^{(n)}(\pi)$ and attempt to spot a pattern.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\sin(x)$	0
1	$\cos(x)$	-1
2	$-\sin(x)$	0
3	$-\cos(x)$	1
4	$\sin(x)$	0
5	$\cos(x)$	-1
6	$-\sin(x)$	0
7	$-\cos(x)$	1
$\vdots$	$\vdots$	$\vdots$

It is not hard to see that when n is even, $f^{(n)}(\pi) = 0$, and when n is odd, $f^{(n)}(\pi)$ is alternately -1 and 1, starting with -1 when $n = 1$, *i.e.* $f^{(2k+1)}(\pi) = (-1)^{k+1}$ for $k \geq 0$. Plugging all this into Taylor's formula, the Taylor series centred at π of $\sin(x)$ is:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-\pi)^n &= \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(\pi)}{(2k+1)!}(x-\pi)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(x-\pi)^{2k+1} \\ &= -(x-\pi) + \frac{(x-\pi)^3}{6} - \frac{(x-\pi)^5}{120} + \cdots\end{aligned}$$

To determine the radius and interval of convergence of this Talor series, we haul out the Ratio Test again.

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{\frac{(-1)^{(k+1)+1}}{(2(k+1)+1)!}(x-\pi)^{2(k+1)+1}}{\frac{(-1)^{k+1}}{(2k+1)!}(x-\pi)^{2k+1}}\right| &= \lim_{k\to\infty}\left|\frac{(-1)^{k+3}(x-\pi)^{2k+3}}{(2k+3)!}\cdot\frac{(2k+1)!}{(-1)^{k+1}(x-\pi)^{2k+1}}\right| \\ &= \lim_{k\to\infty}\left|\frac{(-1)^2(x-\pi)^2}{(2k+3)(2k+2)}\right| \\ &= \lim_{k\to\infty}\frac{(x-\pi)^2}{(2k+3)(2k+2)} \begin{matrix}\to (x-\pi)^2 \\ \to \infty\end{matrix} = 0\end{aligned}$$

Since $0 < 1$ it follows by the Ratio Test that the series converges for all x; that is, its radius of convergence is $R = \infty$ and its interval of convergence is $(-\infty, \infty)$. ■

10. Determine whether the series

$$\sum_{n=0}^{\infty}\left[\frac{1}{4n+1}+\frac{1}{4n+2}-\frac{1}{4n+3}-\frac{1}{4n+4}\right]=1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\cdots$$

converges absolutely, converges conditionally, or diverges. If it is convergent, find its sum. *[10]*

Solution. What follows is pretty verbose, mainly because this is probably the most challenging question on this exam. (It was marked pretty generously because of this, too.) Most of the computation of the sum of the series in the latter part of this solution is adapted from a student's* solution, as being better than what I had originally come up with.

We first tackle the problem of whether the given series converges. Since the corresponding series of positive terms is the harmonic series, which diverges, we do not have absolute convergence. Unfortunately, the pattern of alternation in the given series, $++--$, is not the pattern, $+-+-$, required to apply the Alternating Series Test. We work around these problems by rewriting the series by combining terms within each group of four consecutive terms:

$$\begin{aligned}\frac{1}{4n+1}+\frac{1}{4n+2}-\frac{1}{4n+3}-\frac{1}{4n+4}&=\frac{1}{4n+1}-\frac{1}{4n+3}+\frac{1}{4n+2}-\frac{1}{4n+4}\\&=\frac{(4n+3)-(4n+1)}{(4n+1)(4n+3)}+\frac{(4n+2)-(4n+4)}{(4n+2)(4n+4)}\\&=\frac{2}{(4n+1)(4n+3)}+\frac{2}{(4n+2)(4n+4)}\end{aligned}$$

This converts our original series into a series, $\sum_{n=0}^{\infty}\left[\frac{2}{(4n+1)(4n+3)}+\frac{2}{(4n+2)(4n+4)}\right]$, of positive terms. Since every fourth partial sum of the original series will be equal to every second partial sum of the new series, the original series will converge exactly when (and have the same sum as) the new series converges. Note that since the new series is composed entirely of positive terms, it will converge absolutely if it converges at all, which permits us to rearrange the terms as we like, so long as all the terms get used eventually. This, in turn, means that we can split the new series into two series:

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[\frac{1}{4n+1}+\frac{1}{4n+2}-\frac{1}{4n+3}-\frac{1}{4n+4}\right]\\&=\sum_{n=0}^{\infty}\left[\frac{2}{(4n+1)(4n+3)}+\frac{2}{(4n+2)(4n+4)}\right]\\&=\left[\sum_{n=0}^{\infty}\frac{2}{(4n+1)(4n+3)}\right]+\left[\sum_{n=0}^{\infty}\frac{2}{(4n+2)(4n+4)}\right]\end{aligned}$$

* Thank you, Anika! :-)

We will use the Limit Comparison Test to show that each of the two series converges by comparison with $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$, which itself converges by the p-Test since it has $p = 2 > 1$. Since

$$\lim_{n\to\infty} \left| \frac{\frac{2}{(4n+1)(4n+3)}}{\frac{1}{n^2}} \right| = \lim_{n\to\infty} \frac{2}{(4n+1)(4n+3)} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{2n^2}{16n^2+16n+3}$$
$$= \lim_{n\to\infty} \frac{2n^2}{16n^2+16n+3} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2}{16+\frac{16}{n}+\frac{3}{n^2}} = \frac{2}{16+0+0} = \frac{1}{8}$$

and $0 < \frac{1}{8} < \infty$, we have that $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)}$ converges because $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$ does so. Similarly, since

$$\lim_{n\to\infty} \left| \frac{\frac{2}{(4n+2)(4n+4)}}{\frac{1}{n^2}} \right| = \lim_{n\to\infty} \frac{2}{(4n+2)(4n+4)} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{2n^2}{16n^2+24n+8}$$
$$= \lim_{n\to\infty} \frac{2n^2}{16n^2+24n+8} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2}{16+\frac{24}{n}+\frac{8}{n^2}} = \frac{2}{16+0+0} = \frac{1}{8}$$

and $0 < \frac{1}{8} < \infty$, we also have that $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(4n+2)(4n+2)}$ converges because $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$ does so.

Putting all this together, the facts that $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(4n+2)(4n+2)}$ are series of positive terms that converge (absolutely!) means that their rearranged sum, $\displaystyle\sum_{n=0}^{\infty} \left[\frac{2}{(4n+1)(4n+3)} + \frac{2}{(4n+2)(4n+4)} \right]$, converges too. Since this series has every second partial sum equal to every fourth partial sum of the original series,

$$\sum_{n=0}^{\infty} \left[\frac{1}{4n+1} + \frac{1}{4n+2} - \frac{1}{4n+3} - \frac{1}{4n+4} \right],$$

it follows that the original series converges too, and to the same sum that

$$\sum_{n=0}^{\infty} \left[\frac{2}{(4n+1)(4n+3)} + \frac{2}{(4n+2)(4n+4)} \right]$$

does.

This begs the question, what is the sum of the given series? We will find it by first partially undoing some of our manipulations above, and then appealing to what we learned earlier in the course. Here goes:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left[\frac{1}{4n+1}+\frac{1}{4n+2}-\frac{1}{4n+3}-\frac{1}{4n+4}\right]\\
=&\sum_{n=0}^{\infty}\left[\frac{2}{(4n+1)(4n+3)}+\frac{2}{(4n+2)(4n+4)}\right]\\
=&\left[\sum_{n=0}^{\infty}\frac{2}{(4n+1)(4n+3)}\right]+\left[\sum_{n=0}^{\infty}\frac{2}{(4n+2)(4n+4)}\right]\\
=&\left[\sum_{n=0}^{\infty}\left(\frac{1}{4n+1}-\frac{1}{4n+3}\right)\right]+\left[\sum_{n=0}^{\infty}\left(\frac{1}{4n+2}-\frac{1}{4n+4}\right)\right]\\
=&\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right]+\left[\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots\right]\\
=&\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right]+\frac{1}{2}\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right]
\end{aligned}$$

We summed the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$ in question **6** of Assignment #4. To summarize, integrating both sides of $\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=1-x^2+x^4-x^6+\cdots$ and solving for the constant of integration yields $\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$. Plugging $x=1$ into the latter equation then yields:

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\arctan(1)=\frac{\pi}{4}$$

We also summed the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ in Assignmnent #4, in question **6**. Tu summarize, integrating both sides of $\frac{1}{1+x}=\frac{1}{1-(-x)}=1-x+x^2-x^3+\cdots$ and solving for the constant of integration yields $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$. Plugging $x=1$ into this equation the yields:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\ln(1+1)=\ln(2)$$

Putting all of this together gives us:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left[\frac{1}{4n+1}+\frac{1}{4n+2}-\frac{1}{4n+3}-\frac{1}{4n+4}\right] \\
=&1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\cdots \\
=&\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right]+\left[\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots\right] \\
=&\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right]+\frac{1}{2}\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right] \\
=&\frac{\pi}{4}+\frac{1}{2}\ln(2) \qquad \blacksquare
\end{aligned}
$$

[Total = 100]

Part B. . . . is for bonus! If you want to, do one or both of the following problems.

***α*.** Write a poem touching on calculus or mathematics in general. *[1]*

SOLUTION. I think the Danish architect and poet Piet Hein said it best, in his own twisted way, in the following "grook":

Last Things First

Solutions to problems
  are easy to find:
the problem's a great
  contribution.
What is truly an art
  is to wring from your mind
a problem to fit
  a solution.

Way too much research in math seems to work this way . . . :-) ■

***β*.** A certain mathematician once asserted that $1+2+4+8+\cdots=-1$. What did this unfortunate person do to get this equation? *[1]*

SOLUTION. This is another solution adapted from one given by a student† that is superior to my own:

> The unfortunate who finds $\sum_{n=0}^{\infty} 2^n = 1+2+4+8+\cdots$ to be -1 has done one thing to get this answer: A mistake.

Indeed! :-) ■

I HOPE THAT YOU ENJOYED THE COURSE.
ENJOY THE REST OF THE SUMMER!

† Thank you, David! :-)