Taylor Series V – A little cleanup,

or, the mother of all shortcuts in finding Taylor series,

or, the underlying principle to the shortcuts mentioned in Taylor Series II.

Fact: If $f(x)$ is equal to some power series centred at a,

ie $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$,

then that series is the Taylor series of $f(x)$ centred at a.

Why? This is a consequence of how we got Taylor's formula in the first place, where we assumed $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ and proceeded to show that $a_n = \frac{f^{(n)}(a)}{n!}$.

Cheap example (unlike any of those in Taylor Series II). 3

Suppose we want the Taylor series for $e^{-x^2/2}$.
(This the key part of the function for the normal distribution in statistics and it has no nice antiderivative.)

We do know that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so we simply substitute $-x^2/2$ for x on both sides

$$e^{-x^2/2} = 1 + \frac{-x^2/2}{1!} + \frac{(-x^2/2)^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n \cdot n!}$$

Since this series equals $e^{-x^2/2}$, it follows that it is the Taylor series of $e^{-x^2/2}$ (at 0).

It's been fun! Good luck on the last quiz tomorrow and on the exam!