Series VII - Absolute vs Conditional Convergence

(§11.6 in the text) or, conditional convergence can get weird.

Recall: A series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges, and it converges conditionally if it converges but $\sum_{n=0}^{\infty} |a_n|$ diverges.

Fact: If $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ also convergences.

eg $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges because $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges

[By the p-Test, since $p = 2 > 1$ in this case.]

Note: this is easier as a way to check that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges than using the Alternating Series Test.

What's wrong with conditional convergence?

Note that any finite sum will give you the same answer no matter what order you add up the terms in.

eg $\quad 1+3-4+5 = 5$

and $\quad 5-4+1+3 = 5$

With absolutely convergent series you can scramble infinitely many terms and still get the same sum.

ie As long as you do add up all the terms you will get the same sum no matter what order you do it in.

Fact: A conditionally convergent can be made to add up to any sum we like by rearranging it.

eg Consider the alternating harmonic series (which is the prototype of conditionally convergent series).

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Pick a number, any number, ... Say 2.
We'll make this add up to 2 by rearranging it: Add up enough positive terms to get to or over 2, and then add up enough negative terms to get below 2 again.

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} \quad > 2$$

$$- \frac{1}{2} \quad < 2$$

$$+ \frac{1}{19} + \frac{1}{21} + \frac{1}{23} + \cdots + \frac{1}{2k+1} \quad > 2$$

$$- \frac{1}{4} \quad < 2$$

and so on.

The key here is that the series of the positive terms alone

$$\sum_{k=0}^{\infty} \frac{(-1)^{2k}}{2k+1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$$

actually diverges. Since all the terms are positive it can only diverge by adding up to infinity.

It diverges by the Integral Test:

$$\int_0^{\infty} \frac{1}{2x+1}\,dx = \lim_{a\to\infty} \int_0^{a} \frac{1}{2x+1}\,dx \qquad \begin{aligned} u &= 2x+1 \\ du &= 2\,dx \\ dx &= \tfrac{1}{2}\,du \end{aligned}$$

$$= \lim_{a\to\infty} \int_1^{2a+1} \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\lim_{a\to\infty} \ln(u)\Big|_1^{2a+1} \qquad \begin{array}{c|c} x & u \\ \hline 0 & 1 \\ a & 2a+1 \end{array}$$

$$= \frac{1}{2}\lim_{a\to\infty}\Big[\ln\big(\underbrace{2a+1}_{\downarrow\,\infty}\big) - \underbrace{\ln(1)}_{=\,0}\Big] = \infty.$$

So no matter how many positive terms, we've used, the remaining ones add up to ∞, so we can always get back above 2.

Over infinitely many steps we will eventually use every positive term and every negative term.

But this process guarantees that the partial sums get closer & closer to 2 if you go out far enough, hence the sum of the series rearranged to look like

$$1+\frac{1}{3}+\cdots+\frac{1}{17}-\frac{1}{2}+\frac{1}{19}+\frac{1}{21}+\cdots+\frac{1}{2k+1}-\frac{1}{4}+\frac{1}{2k+3}+\cdots$$

adds up to 2.

Thus conditionally convergent series are a little twitchy: you can only rearrange finitely many terms safely.

Next time: the Ratio and Root Tests

& then power series.