Series V - The Limit Comparison Test

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[Not in the textbook.]

Recall: The (Basic) Comparison Test

Suppose $0 \le a_n \le b_n$ for all n past some point.

(1) If $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(2) If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.

eg Suppose we want to determine whether

$$\sum_{n=0}^{\infty} \frac{n+3^n}{n^2+4^n}$$ dominant terms

converges or not.

Look at the dominant terms in the numerator & denominator and the series ought to converge (or not) as the series $\sum_{n=0}^{\infty} \frac{3^n}{4^n}$. With the Basic Comparison Test we can't just compare to the simplified series because we have to check that it's bigger and it might not be. How do you check if

$$\underset{\substack{\uparrow\\ \text{easy}}}{0 \leq} \frac{n+3^n}{n^2+4^n} \underset{\substack{\uparrow\\ ?}}{\leq} \frac{3^n}{4^n}$$

$n+3^n > 3^n$ not helpful

$n^2+4^n > 4^n$ helpful

since $\frac{1}{n^2+4^n} < \frac{1}{4^n}$

Work-around:

$$0 \leq \frac{n+3^n}{n^2+4^n} \leq \frac{3^n+3^n}{4^n} = 2 \cdot \frac{3^n}{4^n}$$

& $\sum_{n=0}^{\infty} 2 \cdot \frac{3^n}{4^n}$ converges because it is a geometric series with common ratio $|r| = \frac{3}{4} < 1$.

Limit Comparison Test

Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ which are both eventually positive (only finitely many 0s and $a_n > 0$ otherwise, & similarly for b_n).

1) If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a real number > 0,

then both series converge or both diverge.

2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then (a) if $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$;

& (b) if $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$.

3) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then (a) if $\sum_{n=0}^{\infty} a_n$ coneges, so does $\sum_{n=0}^{\infty} b_n$;

& (b) if $\sum_{n=0}^{\infty} b_n$ diverges, so does $\sum_{n=0}^{\infty} a_n$.

eg Another look at $\sum_{n=0}^{\infty} \frac{n+3^n}{n^2+4^n}$ using the Limit Comparison Test. Look for the dominant terms: 3^n in the numerator & 4^n is the denominator, so we'll compare to the simplified series

$$\sum_{n=0}^{\infty} \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$$

[which is geometric with common ratio $|r| = \frac{3}{4} < 1$, so it converges].

$$\lim_{n\to\infty} \frac{\frac{n+3^n}{n^2+4^n}}{\frac{3^n}{4^n}} = \lim_{n\to\infty} \frac{n+3^n}{n^2+4^n} \cdot \frac{4^n}{3^n} = \lim_{n\to\infty} \frac{n+3^n}{n^2+4^n} \cdot \frac{\frac{1}{3^n}}{\frac{1}{4^n}}$$

$$= \lim_{n\to\infty} \frac{\frac{n}{3^n} + \frac{3^n}{3^n}}{\frac{n^2}{4^n} + \frac{4^n}{4^n}} = \lim_{n\to\infty} \frac{\overbrace{\frac{n}{3^n}}^{\to 0} + 1}{\underbrace{\frac{n^2}{4^n}}_{\to 0} + 1} = \frac{0+1}{0+1} = 1 > 0,$$

so the two series both converge or both diverge by the Limit Comparison Test. Since $\sum_{n=0}^{\infty} \frac{3^n}{4^n}$ converges, so does $\sum_{n=0}^{\infty} \frac{n+3^n}{n^2+4^n}$.

eg $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+3n+41}{n^4+n^3-2n^2+n-43}$

This ought to converge: the dominant terms are n^2 in the numerator and n^4 in the denominator, and $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p-Test because $p = 2 > 1$. Using the (Basic) Comparison Test means jumping through a lot of algebraic hoops. With the Limit Comparison Test we just compare the two series directly:

$$\lim_{n\to\infty} \frac{\frac{n^2+3n+41}{n^4+n^3-2n^2+n-43}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2+3n+41}{n^4+n^3-2n^2+n-43} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^4+3n^3+41n^2}{n^4+n^3-2n^2+n-43} \cdot \frac{\frac{1}{n^4}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{1+\underbrace{\frac{3}{n}}_{\to 0}+\underbrace{\frac{41}{n^2}}_{\to 0}}{1+\underbrace{\frac{1}{n}}_{\to 0}-\underbrace{\frac{2}{n^2}}_{\to 0}+\underbrace{\frac{1}{n^3}}_{\to 0}-\underbrace{\frac{43}{n^4}}_{\to 0}}$$

$$= \frac{1+0}{1+0} = 1 > 0$$ so the Limit Comparison Test

tells us that since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges,

so does the given series $\sum_{n=0}^{\infty} \frac{n^2+3n+41}{n^4+n^3+2n^2+n-43}$.

Moral: Easier set-up & algebra (usually) than the (Basic) Comparison Test, at the cost of computing a limit (usually a tedious but not too hard one).

Exercise: Do the homework problems in §11.5 both with the Basic Comparison Test and the Limit Comparison Test.

(To get experience as to which is more likely to be useful in a given situation.)