Series II - Series we can easily sum or tell we can't ①

(An expansion on some things in §11.2 or that ought to be there.)

Recap: A series is a sum of an infinite sequence $\{a_n\}_{n=k}^{\infty}$, written as $\sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \cdots$.

The series converges if the sum makes sense

ie $\lim_{m\to\infty} \sum_{n=k}^{m} a_n = \lim_{m\to\infty} (a_k + a_{k+1} + \cdots + a_m)$

exists (& that limit is the sum of the series).

If the series does not converge, it is said to diverge.

0° Divergence Test If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=k}^{\infty} a_n$ diverges.

Caveat: There are series with $\lim_{n\to\infty} a_n$ that diverge, such as the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, so a series that survives the Divergence Test is not guaranteed to converge.

1° Geometric Series

A geometric series is one with first term a and common ratio r, ie $a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$.
This will have a sum (ie converge) if either $|r|<1$ or $a=0$, and will diverge otherwise.

Why? $(a+ar+ar^2+\cdots+ar^m)(1-r)$

$$= a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^m}$$
$$\quad - \cancel{ar} - \cancel{ar^2} - \cdots - \cancel{ar^m} - ar^{m+1} = a - ar^{m+1}$$

$\sum_{n=0}^{m} ar^n$ Thus $a+ar+ar^2+\cdots+ar^m = \frac{a-ar^{m+1}}{1-r} = \frac{a(1-r^{m+1})}{1-r}$.

It follows that $\sum_{n=0}^{\infty} ar^n$ converges if $\lim_{m\to\infty} \sum_{n=0}^{m} ar^n$

$$= \lim_{m\to\infty} a\frac{(1-r^{m+1})}{(1-r)}$$

$= 0$ if $a=0$ & $= \frac{a}{1-r}$ if $|r|<1$ (so $r^{m+1} \to 0$ as $m\to\infty$)
and does not exist otherwise (as $r^{m+1} \to \infty$ if $|r|>0$,

$r^{n+1} \to$ DNE if $r = -1$, (3)

& we'd be dividing by 0 if $r = 1$.)

es $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$

$= x - x^3 + x^5 - x^7 + \cdots$

is a geometric series with $a = x$ & $r = -x^2$,
so it converges to $\frac{x}{1-(-x^2)} = \frac{x}{1+x^2}$ when $|x| < 1$
and diverges otherwise.

[We are headed towards dealing with "power series"

$\sum_{n=0}^{\infty} a_n x^n$ and writing functions in terms of such series.]

$n! = n(n-1)(n-2)\cdots 2\cdot 1$ es e^{x^2} has no nice antiderivative but since
$(0! = 1)$ $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$, we can write

$\int e^{x^2} dx$ as $\int \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \right) dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\cdot n!}$

2° <u>Telescoping series</u> – series in which successive terms cancel each other out

eg $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$

Does this converge; if so, what to?

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, so this series comes apart

as $\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$

Thus $\sum_{n=1}^{m} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{m}} - \frac{1}{m+1}\right)$

$= 1 - \frac{1}{m}$, so

$\lim_{m \to \infty} \sum_{n=1}^{m} \frac{1}{n(n+1)} = \lim_{m \to \infty} \left(1 - \underbrace{\frac{1}{m}}_{0}\right) = 1 - 0 = 1$

$\therefore \sum_{n=0}^{\infty} \frac{1}{n(n+1)}$ converges and sums to 1.

3° (Pro tip.) Note that $\sum_{n=k}^{\infty} a_n$ converges or diverges

exactly as $\sum_{n=m}^{\infty} a_n$ converges or diverges.

$(k<m)$

$$\lim_{b\to\infty} \sum_{n=k}^{b} a_n = \lim_{b\to\infty}\left[(a_k + a_{k+1} + \cdots + a_{m-1}) + \sum_{n=m}^{b} a_n\right]$$

$$= (a_k + \cdots + a_{m-1}) + \lim_{b\to\infty}\sum_{n=m}^{b} a_n$$

so the limit exists if and only if limit exists,

i.e. one series converges (or diverges)

if and only if the other does.

eg $\sum_{n=10}^{\infty} \frac{1}{n}$ diverges because $\sum_{n=1}^{\infty} \frac{1}{n}$ (the harmonic series)

$= \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots$ diverges.