Series – the basics (§11.2)

Def'n: Given a sequence $\{a_n\}$ ($n \geq k$), the corresponding series is the infinite sum

$$a_k + a_{k+1} + a_{k+2} + \cdots = \underbrace{\sum_{n=k}^{\infty} a_n}_{\text{standard notation.}}$$

Warning: Not every infinite sum makes sense.

eg $a_n = 1$ ($n \geq 0$)

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} 1 = 1+1+1+1+\cdots$$

"adds up" to ∞, which doesn't count as a real number.

eg $a_n = (-1)^n$ ($n \geq 0$)

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+1-1+\cdots$$

doesn't add up because the sum ought to be 1 or 0 (or perhaps -1) but can't tell which.

What does it mean for a series (ie an attempted infinit sum) to actually add up to a real number?

<u>Def'n</u>: The <u>partial sums</u> of the series $\sum_{n=k}^{\infty} a_n$

are the finite sums

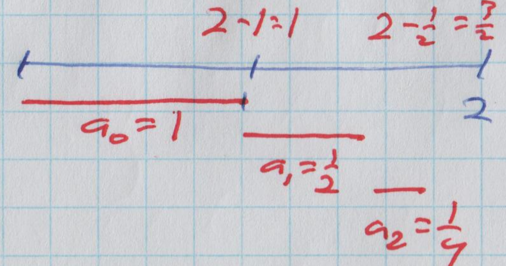
$$s_m = \sum_{n=k}^{m} a_n = a_k + a_{k+1} + a_{k+2} + \cdots + a_m .$$

The series $\sum_{n=k}^{\infty} a_n$ "adds up" to a real number s (usually said as it "<u>converges</u>") if

$$\lim_{m\to\infty} s_m \text{ exists and } = s .$$

<u>eg</u> $a_n = \frac{1}{2^n}$ $(n \geq 0)$, so $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

If $m \geq 0$, $s_m = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^m} = 2 - \frac{1}{2^m}$.



$1 + \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4}$ ... .

Thus, since $\lim_{m\to\infty} s_m = \lim_{m\to\infty}\left(\underset{\downarrow\atop 2}{2} - \underset{\downarrow\atop 0}{\frac{1}{2^m}}\right) = 2 - 0 = 2$

exists and equals 2, we can say that

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

If you have a sequence where $a_n = ar^n$ $(n \geq 0)$,

geometric series.

then $a + ar + ar^2 + \cdots + ar^m = s_m$

$$= a(1 + r + r^2 + \cdots + r^m)$$

$$= a \cdot \frac{1 - r^{m+1}}{1 - r}.$$

(Sum of finite geometric series.)

Terminology: A series that adds up is said to converge & one that does not add up is said to diverge.

$\Rightarrow$ $\left\{\frac{1}{2^n}\right\}$ gives us a series, $\sum_{n=0}^{\infty} \frac{1}{2^n}$, that converges.

but $\{2^n\}$ — " —, $\sum_{n=0}^{\infty} 2^n = 1+2+4+8+\cdots$,

that diverges:

$$s_m = \underbrace{1+2+4+\cdots+2^m}_{a \,\cdot 2\ \cdot 2\ \cdot 2} = 2^{m+1}-1$$

so this a geometric series with "common ratio" $r=2$

and hence partial sum $s_m = 1+2+\cdots+2^m$

$$= 1\cdot\frac{1-2^{m+1}}{1-2}$$

$$= \frac{1-2^{m+1}}{-1} = 2^{m+1}-1$$

Since $s_m = 2^{m+1}-1 \to \infty$ as $m \to \infty$,

the series $\sum_{n=0}^{\infty} 2^m$ diverges.

We'll develop a suite of tests that will tell us whether various series we encounter converge or diverge. (No test will work for all series, so we need lots of tests...)

Basic Properties

1) If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge, so does $\sum_{n=0}^{\infty} (a_n + b_n)$ and moreover

$$\sum_{n=0}^{\infty} (a_n + b_n) = \left(\sum_{n=0}^{\infty} a_n\right) + \left(\sum_{n=0}^{\infty} b_n\right).$$

[Why? Suppose $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$.

ie $\lim_{m\to\infty} \left(\sum_{n=0}^{m} a_n\right) = A$ and $\lim_{m\to\infty} \left(\sum_{n=0}^{m} b_n\right) = B$,

so $\lim_{m\to\infty} \sum_{n=0}^{m} (a_n + b_n) = \lim_{m\to\infty} \left(\left(\sum_{n=0}^{m} a_n\right) + \left(\sum_{n=0}^{m} b_n\right)\right)$

$= \left[\lim_{m\to\infty} \left(\sum_{n=0}^{m} a_n\right)\right] + \left[\lim_{m\to\infty} \left(\sum_{n=0}^{m} b_n\right)\right] = A + B.$]

2) If $\sum_{n=0}^{\infty} a_n$ converges and c is any constant,

then $\sum_{n=0}^{\infty} ca_n$ converges and $\sum_{n=0}^{\infty} ca_n = c\left(\sum_{n=0}^{\infty} a_n\right)$.

<u>Caveats</u>: You really need the hypotheses in these rules.

<u>eg</u> 1) $a_n = n$ & $b_n = -n$

$\sum_{n=0}^{\infty} n$ and $\sum_{n=0}^{\infty} b_n$ both diverge,

but $\sum_{n=0}^{\infty} (n + (-n)) = \sum_{n=0}^{\infty} 0$ converges.

2) $a_n = n$ & $c = 0$

$\sum_{n=0}^{\infty} n$ diverges but $\sum_{n=0}^{\infty} ca_n = \sum_{n=0}^{\infty} 0$ converges.

3) If $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \to 0$.

(ie $\lim_{n\to\infty} a_n = 0$).

This fact usually gets used in reverse:

<u>Divergence Theorem:</u> If $\lim_{n\to\infty} a_n \neq 0$,

then $\sum_{n=0}^{\infty} a_n$ diverges.

<u>Warning:</u> You can have sequences $\{a_n\}$ such that $a_n \to 0$ but $\sum_{n=0}^{\infty} a_n$ diverges.

$\underrightarrow{\text{eg}}$ The <u>Harmonic Series</u>: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Obviously, $\lim_{n\to\infty} \frac{1}{n} = 0$.

However the series diverges:

$$\sum_{n=1}^{\infty}\frac{1}{n} = \underbrace{1}_{\substack{\geq \\ 1}} + \underbrace{\frac{1}{2}+\frac{1}{3}}_{\substack{\geq \\ 2\cdot\frac{1}{3} \\ \geq \\ 2\cdot\frac{1}{4} \\ = \\ \frac{1}{2}}} + \underbrace{\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}}_{\substack{\geq \\ 4\cdot\frac{1}{7} \\ \geq \\ 4\cdot\frac{1}{8} \\ = \\ \frac{1}{2}}} + \underbrace{\frac{1}{8}+\frac{1}{9}+\cdots}_{\substack{\geq \\ 8\cdot\frac{1}{15} \\ \geq \\ 8\cdot\frac{1}{16} \\ = \\ \frac{1}{2}}} \cdots$$

So adding up $\sum_{n=1}^{\infty}\frac{1}{n}$ is like adding up

$$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+ \cdots$$

$$= 1+\sum_{n=2}^{\infty}\frac{1}{2}$$

which diverges by the Divergence Theorem because $\lim_{n\to\infty}\frac{1}{2} = \frac{1}{2} \neq 0$.