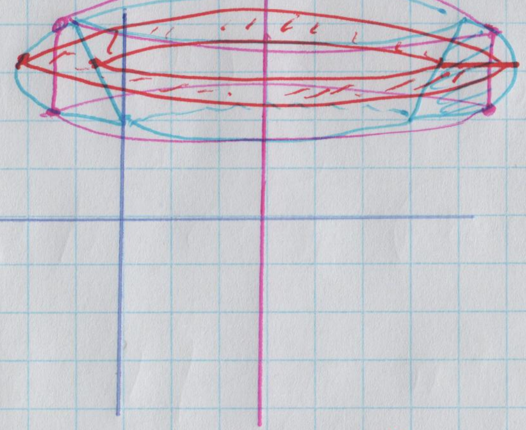
Volumes III - solids of revolution continued.

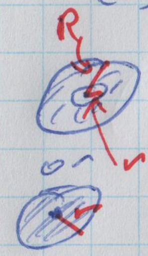
Recap: A solid of revolution is what you get when you revolve a 2-D region about a line ("axis of revolution") that does not pass inside the region, though it may touch the border of the region.

Tip: sketch the region and the solid to help with setting up the volume integral,



Two methods:

(1) Disk/washer method:



cross-sections perpendicular to the axis of revolution.

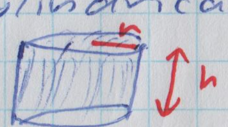
Use πr^2 formula for the area of a circle.

dish: $A = \pi r^2$

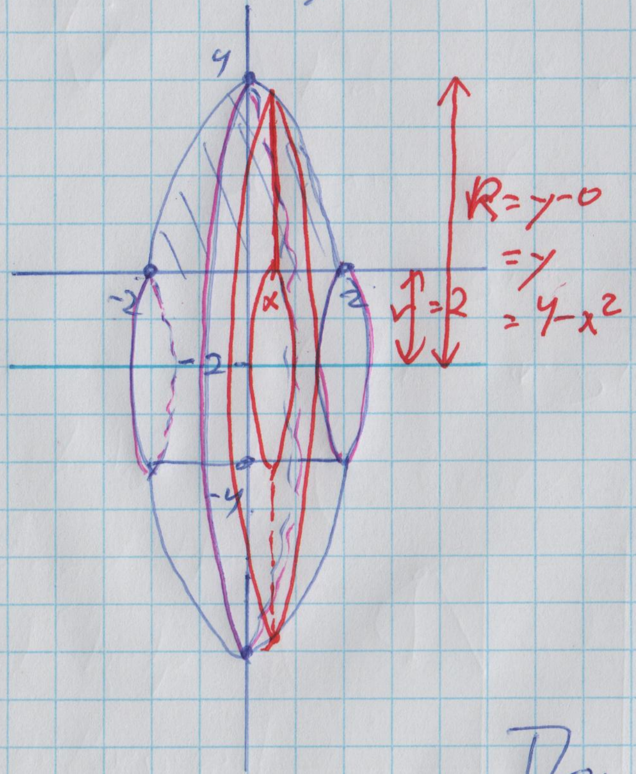
washer: $A = \pi R^2 - \pi r^2$

In both integrate area, to get volume...

(2) Cylindrical shell method: Use the area formula for a cylinder without the caps: $A = 2\pi r h$



eg Suppose our region is the one below $y = 4 - x^2$ and above $y = 0$. (so $-2 \le x \le 2$)



First, let's revolve this about the line $y = -2$.

(1) If we use the disk/washer method, then we use x as the variable, since the x-axis is perpendicular to the cross-section (ie parallel to the axis of revolution).

Then the cross-section at x has outer radius $R = y - 0 = 4 - x^2$ and inner radius $r = 0 - (-2) = 2$.

$$V = \int_{-2}^{2} (\pi R^2 - \pi r^2)\,dx = \pi \int_{-2}^{2} \left((4-x^2)^2 - 2^2 \right) dx = \pi \int_{-2}^{2} \left[4^2 - 2\cdot 4x^2 + (x^2)^2 - 4 \right] dx$$

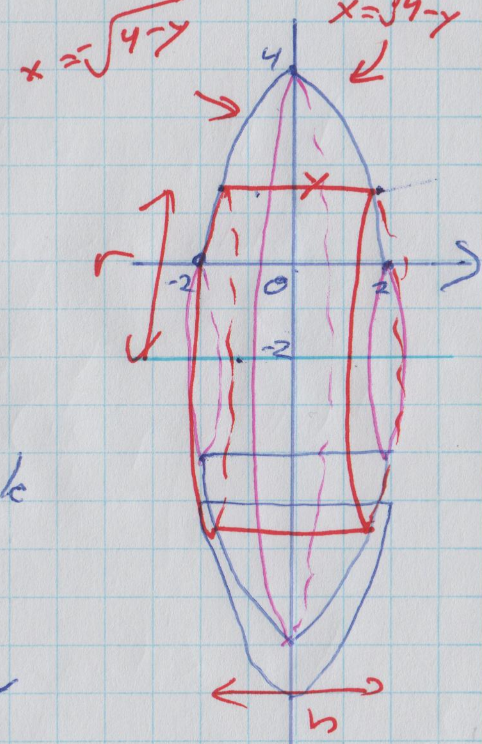
$$= \pi \int_{-2}^{2} (12 - 8x^2 + x^4)\,dx = \pi \left(12x - \frac{8}{3}x^3 + \frac{x^5}{5} \right) \Big|_{-2}^{2}$$

$$= \pi\left(12\cdot 2 - \frac{8}{3}\cdot 2^3 + \frac{2^5}{5}\right) - \pi\left(12\cdot(-2) - \frac{8}{3}(-2)^3 + \frac{(-2)^5}{5}\right)$$

$$= \pi\left(24 - \frac{64}{3} + \frac{32}{5}\right) - \pi\left(-24 - \frac{8}{3}(-8) + \frac{-32}{5}\right)$$

$$= 24\pi - \cancel{\frac{64}{3}\pi} + \frac{32}{5}\pi + 24\pi + \cancel{\frac{64}{3}\pi} + \frac{32}{5}\pi$$

$$= 48\pi + \frac{64}{5}\pi = \frac{240}{5}\pi + \frac{64}{5}\pi = \frac{304}{5}\pi$$



2. What about the cylindrical shell method?

In this case we should use y as the variable since the y-axis is perpendicular to the shell.

In terms of y, the region is given by

$$-\sqrt{4-y} \le x \le \sqrt{4-y} \quad \text{for } 0 \le y \le 4.$$

$$y = 4 - x^2$$
$$\Rightarrow x^2 = 4 - y$$
$$\Rightarrow x = \pm\sqrt{4-y}$$

The shell at y has radius $r = \sqrt{4-y} - (-\sqrt{4-y})$
$$= 2\sqrt{4-y}$$

and height $h = y - (-2) = y + 2$.

This gives us the volume integral

$$V = \int_0^4 2\pi r h \, dy = 2\pi \int_0^4 2\sqrt{4-y} \cdot (y+2) \, dy$$

$u = 4-y$
$du = -dy$
$dy = (-1)du$

y	u
0	4
4	0

$y = 4-u$

$$= 4\int_4^0 \sqrt{u}\,(\cancel{4}^{2} - u \cancel{+2})(-1)\,du$$

$$= -4\int_4^0 (2u^{1/2} - u^{3/2})\,du = 4\int_0^4 (2u^{1/2} - u^{3/2})\,du$$

$$= 4\left(\frac{2u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right)\Bigg|_0^4 = \left(\frac{16}{3}u^{3/2} - \frac{8}{5}u^{5/2}\right)\Bigg|_0^4$$

$4^{1/2} = 2$
$4^{3/2} = 2^3 = 8$
$4^{5/2} = 2^5 = 32$

$$= \left(\frac{16}{3}\cdot 8 - \frac{8}{5}\cdot 32\right) - \cancel{\left(\frac{16}{3}\cdot 0 - \frac{8}{5}\cdot 0\right)}$$

$$= \frac{128}{3} - \frac{256}{5} = \ldots$$

Put it all over the common denominator of 15, if you care!

Exercise: Try to revolve this same region about the line $x = 3$ and do it both ways.

Note: With two methods in play, one or the other may be easier in a given situation. Practice so that you can figure out which is easier faster. Worst case: set it up both ways and do the integral that looks easier.