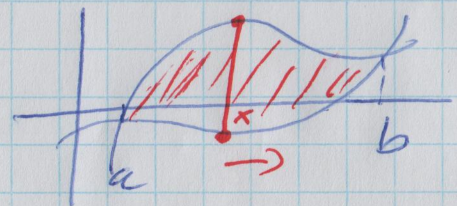
Volumes (of 3-D regions with nice cross-sections)

[§9.3 in the textbook - please skim through §9.2, which is about position/velocity/acceleration on your own.]

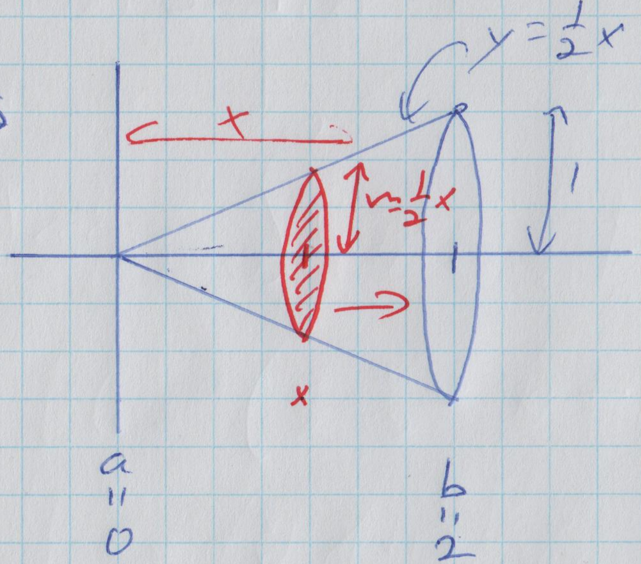
For areas, cross-sections swept out areas.

eg

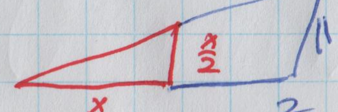


For volumes, (areas of) cross-sections sweep out volume.

eg



Suppose we have cone on its side with its tip at the origin and its axis of symmetry being the x-axis. The cone has length 2 & radius at the blunt end 1.



Similar triangles tell us that the radius of the circular cross-section at x is $\frac{x}{2}$.

~~Area~~ Volume of the cone $= \int_a^b (\text{area of cross-section at } x)\, dx$

$$A(x) = \pi r^2 = \pi \left(\frac{x}{2}\right)^2 = \frac{\pi}{4} x^2$$

$$= \int_0^2 \frac{\pi}{4} x^2\, dx = \frac{\pi}{4} \cdot \frac{x^3}{\cancel{12}\,3}\Big|_0^2 = \frac{\pi}{4} \cdot \frac{8}{\cancel{12}\,3} - \cancel{\frac{\pi}{4} \cdot \frac{0}{12\,3}}$$

$$= \frac{\pi \cdot 2}{\cancel{12}\,3} = \frac{2\pi}{3}.$$

The formula for the area of a right circular cone is $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi \cdot 1^2 \cdot 2 = \frac{2}{3}\pi \neq \frac{\pi}{6}$

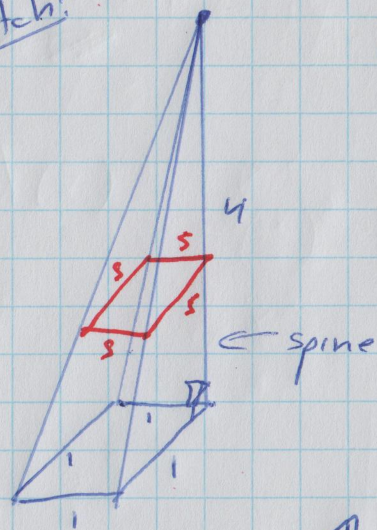
radius at the blunt end

height

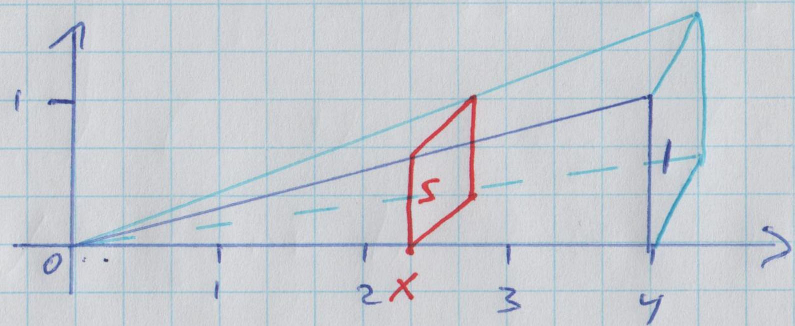
Example: A prism has a base that is square (with sides of length 1) and a top that is a single point that is 4 units above one corner of the base.

What is it's volume?

Sketch:



Cross-sections parallel to the base are also squares that shrink as you go up. We'll place the prism with it's spine along the x-axis and the tip at the origin to help set up the integral:



The cross-section at x (similar triangles) has a side-length s that is to x as 1 is to 4.

ie $\frac{s}{x} = \frac{1}{4} \Rightarrow s = \frac{x}{4}$

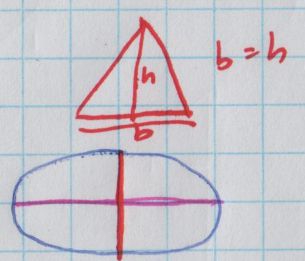
Thus the area of the cross-section at x is $A(x) = s^2 = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$, and hence the volume of the prism is

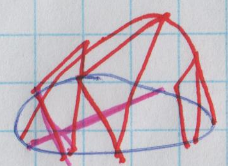
$$V = \int_0^4 A(x)\,dx = \int_0^4 \frac{x^2}{16}\,dx = \frac{1}{16}\cdot\frac{x^3}{3}\Big|_0^4 = \frac{x^3}{48}\Big|_0^4 = \frac{4^3}{48} - \frac{0^3}{48}$$

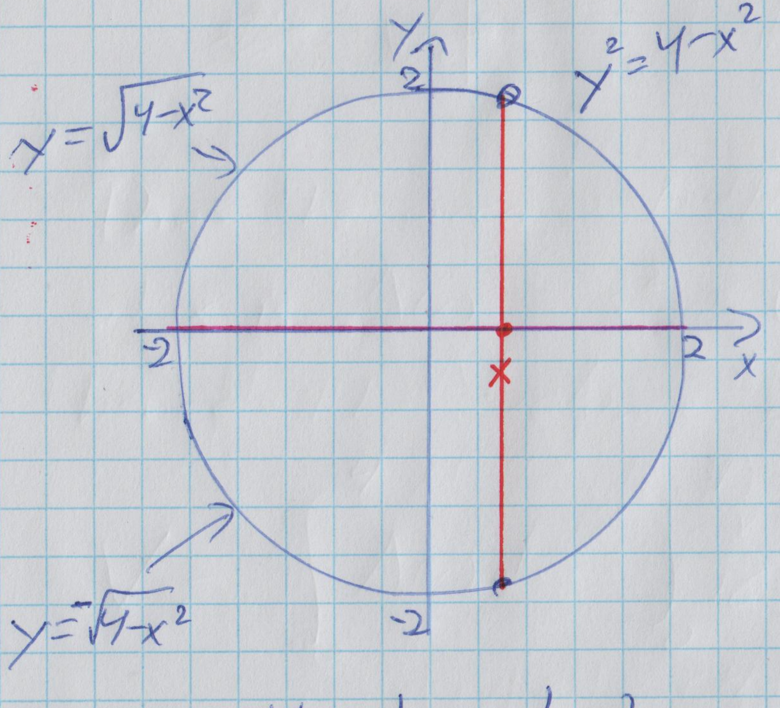
$$= \frac{64}{48} - 0 = \frac{4\cdot\cancel{16}}{3\cdot\cancel{16}} = \frac{4}{3}.$$

Example: A hat has a circular base and the cross-sections perpendicular to one particular diameter are triangles with heights equal to their bases. What is the volume of the region inside the hat? (Assuming the circular base has radius 2 & hence diameter 4.)

Sketch:







We'll lay the base down on the Cartesian plane so it's center is at the origin and the particular diameter is along the x-axis.

The base of the triangular cross-section at x has length $\sqrt{4-x^2}-(-\sqrt{4-x^2})=2\sqrt{4-x^2}$.

This is equal to the height of the triangle so the triangle has area $A(x)=\frac{1}{2}bh=\frac{1}{\cancel{2}}\cancel{2}\sqrt{4-x^2}\cdot 2\sqrt{4-x^2}=2(4-x^2)$

So the volume enclosed by the hat is

$$V = \int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} 2(4-x^2)\,dx = \int_{-2}^{2} (8-2x^2)\,dx$$

$$= \left(8x - \frac{2}{3}x^3\right)\Big|_{-2}^{2} = \left(8\cdot 2 - \frac{2}{3}\cdot 2^3\right) - \left(8(-2) - \frac{2}{3}(-2)^3\right)$$

$$= \left(16 - \frac{16}{3}\right) + \left(+16 + \frac{-16}{3}\right) = \left(16 - \frac{16}{3}\right) + \left(16 - \frac{16}{3}\right)$$

$$= 32 - \frac{32}{3} = \frac{96}{3} - \frac{32}{3} = \frac{64}{3} \quad (\text{units}^3)$$

Next time: Volumes of solids of revolution.