Integrating Rational Functions (§8.5 in the text)

2021-06-28 ①

(a.k.a. the technique of partial fractions)

A rational function is a ratio of polynomials,

eg $f(x) = \frac{x^5 - 2x + 2}{x^4 - 1}$.

How do we (in general) integrate such functions?

0° Make the leading coefficients of the polynomials equal to 1.

eg $g(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0}$

$= \frac{a_n}{b_n} \cdot \frac{x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_1}{a_n} + \frac{a_0}{a_n}}{x^k + \frac{b_{k-1}}{b_k} x^{k-1} + \cdots + \frac{b_0}{b_k}}$

In our example, this is already done.

1° Ensure that the degree of the numerator is less than the degree of the denominator.

How? By dividing the denominator into the numerator (if the degrees are not already satisfactory).

eg $x^4-1\overline{)\,x^5+0x^4+0x^3+0\cdot x^2-2x+2}$ with quotient x

$x^5 = x\cdot x^4$

$-[x^5 \qquad\qquad -x]$

$= 0\cdot x^5 \qquad\qquad \boxed{-x+2}$

→ degree less than the degree of x^4-1 so we stop.

Division of polynomials is regular "long division" with the powers representing the places

Thus $x^5-2x+2 = x(x^4-1)-x+2$

$$\int\frac{x^5-2x+2}{x^4-1}\,dx=\int\frac{x(x^4-1)-x+2}{x^4-1}\,dx=\int\frac{x\cancel{(x^4-1)}}{\cancel{x^4-1}}\,dx+\int\frac{-x+2}{x^4-1}\,dx$$

degree of the numerator < degree of the denominator ↓

2° Factor the denominator fully,

ie into linear factors and/or irreducible quadratic factors,

and collect like terms into powers of those factors.

eg
$$x^4-1 = (x^2)^2-1 = \overset{\substack{\text{irreducible [no roots]}\\ \downarrow}}{(x^2+1)}(x^2-1)$$
$$= (x^2+1)(x-1)(x+1)$$

⌈If you had several copies of $(x-1)$, say, collect them into a power

eg $(x-1)(x-1)(x-1) = (x-1)^3$. ⌋

This is very hard to do normally. The quadratic formula will tell you if a quadratic can be factored; there are formulas that will do this for cubic & quartic polynomials, but past degree ~~5~~4 (ie 5 & up) there is no general formula that will help you factor...

In practice, try to hand off higher degree polynomials to computer algebra programs to do.

Thus, in our example, we have

$$\int \frac{x^5-2x+2}{x^4-1}\,dx = \int x\,dx + \int \frac{-x+2}{x^4-1}\,dx$$

$$= \frac{x^2}{2} + (-1)\int \frac{x-2}{(x^2+1)(x-1)(x+1)}\,dx$$

3° Rewrite the integrand in terms of "partial fractions"

- see $(x^2+ax+b)^n$ in the denominator, the corresponding sum of partial fractions is (A_i, B_i are unknown constants)

  $$\frac{A_n x + B_n}{(x^2+ax+b)^n} + \frac{A_{n-1}x + B_{n-1}}{(x^2+ax+b)^{n-1}} + \cdots + \frac{A_1 x + B_1}{x^2+ax+b}$$

- see $(x+c)^n$ in the denominator, the corresponding sum of partial fractions is $\frac{A_n}{(x+c)^n} + \frac{A_{n-1}}{(x+c)^{n-1}} + \cdots + \frac{A_1}{(x+c}$

Fact: If you add up all the sums of partial fractions for a given integrand, there will be a unique value for the unknown constants that you can figure out with some algebra.

In our example, $\frac{x-2}{(x^2+1)(x-1)(x+1)}$ ← only one power of each

so it's partial fraction expansion is

$$\frac{x-2}{(x^2+1)(x-1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+1}$$

← put this over a common denominator & equate numerators

$$= \frac{(Ax+B)(x-1)(x+1) + C(x^2+1)(x+1) + D(x^2+1)(x-1)}{(x^2+1)(x-1)(x+1)}.$$

so $x-2 = (Ax+B)(x^2-1) + C(x^3+x^2+x+1) + D(x^3-x^2+x-1)$

$'' \quad = Ax^3 - Ax + Bx^2 - B + Cx^3 + Cx^2 + Cx + C + Dx^3 - Dx^2 + Dx - D$

$0x^3 + 0\cdot x^2 + 1\cdot x - 2 = (A+C+D)x^3 + (B+C-D)x^2 + (-A+C+D)x + (-B+C-D)$

Thus we must have:

① $A \qquad + C + D = 0$

② $\qquad B + C - D = 0$

③ $-A \qquad + C + D = 1$

④ $\qquad -B + C - D = -2$

} system of 4 linear equations in 4 unknowns [Feel free to apply your favourite methods from linear algebra.]

I'll do this in the most primitive way using substitution

① $\Rightarrow A = -C - D$ ⑤

Substitute into ③ to get $-(-C-D) + C + D = 1$

$\Rightarrow C + D + C + D = 1 \Rightarrow C + D = \frac{1}{2}$ [illegible]

$\Rightarrow C = -D + \frac{1}{2}$ ⑥

We have ②, ④, & ⑥ in play.

Substitute into ② $B + (-D + \frac{1}{2}) - D = 0$

$\Rightarrow B = -\frac{1}{2} + 2D$ ⑦

Substitute into ④ $-(-\frac{1}{2} + 2D) + (-D + \frac{1}{2}) - D = 2$

$= \frac{1}{2} - 2D - D + \frac{1}{2} - D \Rightarrow 4D = 2 - 1 = 1$

$\Rightarrow D = \frac{1}{4}$

$\Rightarrow \quad C = -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4}$ from ⑥

$\Rightarrow \quad B = -\frac{1}{2} + 2\left(\frac{1}{4}\right) = 0$ ⑦

$\Rightarrow \quad A = -\left(-\frac{3}{4}\right) - \frac{1}{4} = \frac{1}{2}$ ⑤

$$\therefore \frac{x-2}{(x^2+2)(x-1)(x+1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1} + \frac{D}{x+1}$$

$$= \frac{\frac{1}{2}x}{x^2+2} + \frac{-\frac{3}{4}}{x-1} + \frac{\frac{1}{4}}{x+1},$$

so

$$\int \frac{x^5-2x+2}{x^4-1}\,dx = \frac{x^2}{2} + \frac{1}{2}\int \frac{x}{x^2+2}\,dx - \frac{3}{4}\int \frac{1}{x-1}\,dx + \frac{1}{4}\int \frac{1}{x+1}\,d$$

& now one finish the job with substitutions ...

Moral: We can - in principle - do rational functions but even the humanly doable can take a long long time.