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Trigonometric Integrals III - mixed trig functions

1° $\int \sin^{2}(x)\cos^{3}(x)\,dx$ (even, odd power)

Option 1: apply $\sin^2(x) = 1-\cos^2(x)$ & then use the reduction formula for $\cos(x)$ [works since 2 is even]

Option 2: apply $\cos^2(x) = 1-\sin^2(x)$ & then substitute for $\sin(x)$ (we'll try this one) [works since 3 is odd]

$= \int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx$

$u=\sin(x) \quad du = \cos(x)\,dx$

$= \int u^2(1-u^2)\,du$

$= \int (u^2-u^4)\,du$

$= \frac{u^3}{3} - \frac{u^5}{5} + C \quad = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C$

In general, we have similar options

if $\int \sin^k(x)\cos^n(x)\,dx$ ~~and~~ n is odd.

Similarly if k is odd [use $1-\cos^2(x) = \sin^2(x)$ in option 2 & substitute for $\cos(x)$]

If k is even or n is ~~odd~~ even we can try option 1]

2° $\int \cos^2(x)\sin(x)\,dx = \int (1-\sin^2(x))\sin(x)\,dx$ ②

Using the reduction formula.

$= \int (\sin(x) - \sin^3(x))\,dx = \int \sin(x)\,dx - \int \sin^3(x)\,dx$

$= -\cos(x) - \left[-\frac{1}{3}\sin^2(x)\cos(x) + \frac{2}{3}\int \sin(x)\,dx \right]$

$= -\cos(x) + \frac{1}{3}\sin^2(x)\cos(x) + \frac{2}{3}(+\cos(x)) + C$

$= -\frac{1}{3}\cos(x) + \frac{1}{3}\sin^2(x)\cos(x) + C$

$= \frac{1}{3}\cos(x)\left[-1 + \sin^2(x)\right] + C$

$= +\frac{1}{3}\cos(x)\left(-\cos^2(x)\right) + C$

$= -\frac{1}{3}\cos^3(x) + C$

There is also a reduction formula (rarely used) for mixed $\sin(x)$ & $\cos(x)$; see the summary if you're curious

3° $\int \tan^3(x) \underbrace{\sec^2(x)\,dx}_{du}$

1. Substitute $u = \tan(x)$, so $du = \sec^2(x)\,dx$
2. Use $\sec^2(x) = \tan^2(x) + 1$ to replace the $\sec^2(x)$ and then use the reduction formulas.

In this case option is much easier...

$= \int u^3\,du$

$= \cancel{\int} \frac{u^4}{4} + C$

$= \frac{1}{4}\tan^4(x) + C$

4° $\int \tan^3(x) \overbrace{\sec^4(x)}^{\sec^2(x)\sec^2(x)}\,dx$ replace one $\sec^2(x) = 1 + \tan^2(x)$

$= \int \tan^3(x)\,(1 + \tan^2(x))\,\sec^2(x)\,dx$ ... and then substitute $u = \tan(x)$.

5° $\int \tan^3(x)\sec^3(x)\,dx$

If both powers are odd, isolate one copy of sec(x) tan(x) & the replace the rest of the tan's by using $\tan^2(x) = \sec^2(x) - 1$.

$$= \int \tan^2(x)\sec^2(x)\tan(x)\sec(x)\,dx$$

$$= \int (\sec^2(x) - 1)\sec^2(x)\underbrace{\tan(x)\sec(x)\,dx}_{du}$$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$$= \int (u^2 - 1)u^2\,du$$

$$= \int (u^4 - u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C$$

Notice that in this case, since you can't get to just sec(x) or just tan(x), the reduction formulas are useless.

So the general procedure for mixed powers of $\tan(x)$ and $\sec(x)$ is as follows:

$$\int \tan^k(x)\sec^n(x)\,dx$$

1. If one of k or n is even, we can use $\tan^2(x)+1=\sec^2(x)$ to put it entirely in terms of $\tan(x)$ or $\sec(x)$ & then expand and apply the reduction formulas for $\tan(x)$ or $\sec(x)$.

2. If both are odd, factor out one copy of $\tan(x)\sec(x)$, convert the rest into $\sec(x)$ using $\tan^2(x)=\sec^2(x)-1$ & substitute $u=\sec(x)$ so $du=\tan(x)\sec(x)\,dx$.

3. If n is even, it's probably easier (than 1) to convert $\sec^{n-2}(x)$ into $\tan(x)$ using $\sec^2(x)=1+\tan^2(x)$, and then substitute ~~for~~ $u=\tan(x)$, so $du=\sec^2(x)\,dx$.

What happens with more complicated integrands?

eg 1° $\int \sec^2(x)\sin(x)\cos(x)\,dx$

I usually try to put everything in terms of $\cos(x)$ & $\sin(x)$ & see what I can do.

$$= \int \frac{1}{\cos^2(x)} \sin(x)\cos(x)\,dx$$

$$= \int \frac{\sin(x)}{\cos(x)}\,dx = \int \tan(x)\,dx = -\ln(\cos(x)) + C$$

$$= \ln(\sec(x)) + C$$

2° $$\int \frac{\tan(x) + \sin^2(x)}{\sec^2(x) + \cos(x)}\,dx = \int \frac{\frac{\sin(x)}{\cos(x)} + \sin^2(x)}{\frac{1}{\cos^2(x)} + \cos(x)}\,dx$$

multiply by $\frac{\cos^2(x)}{\cos^2(x)}$

$$= \int \frac{\left[\frac{\sin(x)}{\cos(x)} + \sin^2(x)\right]\cdot\cos^2(x)}{\left[\frac{1}{\cos^2(x)} + \cos(x)\right]\cdot\cos^2(x)}\,dx = \int \frac{\sin(x)\cos(x) + \sin^2(x)\cos^2(x)}{1 + \cos^3(x)}\,dx$$

& now I would give up.

The disturbing moral is that some integrals are very hard to find an antiderivative for.

<u>In some cases, there is no nice antiderivative.</u>

<u>Classic example:</u> the standard normal distribution in statistics has the "density function"

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

To compute probabilities with this you need to evaluate integrals of the form $\int_0^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ which is hard because $g(x)$ has no nice antiderivative.

[You can get one in terms of an infinite series, but that is as good as it gets.]