# Trigonometric Integrals II - doing without the reduction formulas, or trying to get to them.

Lots of things to try - basically, try to use trig identities &/or substitution to help.

The basic trig identities most likely to be useful:

1° $\sin^2(x) + \cos^2(x) = 1$ Often used in the form $\sin^2(x) = 1 - \cos^2(x)$ or $\cos^2(x) = 1 - \sin^2(x)$.

2° $1 + \tan^2(x) = \sec^2(x)$ Often used in the form $\tan^2(x) = 1 - \sec^2(x)$.

3° $\sin(2x) = 2\sin(x)\cos(x)$ [A special case of $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$.]

4° $\cos(2x) = \cos^2(x) - \sin^2(x)$

$= 2\cos^2(x) - 1$

$= 1 - 2\sin^2(x)$

Often used in rearranged form:

$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$

$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$

A special case of

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

examples:

1° $\int \sin^5(x)\,dx = \int \sin^4(x)\sin(x)\,dx$

$= \int \left(\sin^2(x)\right)^2 \sin(x)\,dx$

$= \int \left(1-\cos^2(x)\right)^2 \sin(x)\,dx$

Substitute $u = \cos(x)$, so $du = -\sin(x)dx$ & $\sin(x)dx = (-1)du$

$= \int (1-u^2)^2\,du$

$= \int (1-2u^2+u^4)\,du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$

$= \cos(x) - \frac{2}{3}\cos^3(x) + \frac{1}{5}\cos^5(x) + C$

... this is probably faster than using the reduction formulas.

This kind of trick works fine for odd powers of sin or cos's but does not work so well for even powers. For even powers we can try:
if at all

$2^\circ$ $\int \sin^4(x)\,dx = \int (\sin^2(x))^2\,dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx$

$$= \frac{1}{4}\int (1-\cos(2x))^2\,dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \cos^2(2x)\right) dx$$

$$= \frac{1}{4}\int 1\,dx - \frac{2}{4}\int \cos(2x)\,dx + \frac{1}{4}\int \cos^2(2x)\,dx$$

$u = 2x$
$du = 2dx \Rightarrow dx = \frac{1}{2}du$

$$= \frac{1}{4}x - \frac{1}{2}\int \cos(u)\cdot\frac{1}{2}du + \frac{1}{4}\int \left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right) dx$$

$$= \frac{x}{4} - \frac{1}{4}\sin(u) + \frac{1}{8}\int 1\,dx + \frac{1}{8}\int \cos(4x)\,dx$$

$w = 4x$, so
$dw = 4dx$
& $dx = \frac{1}{4}dw$

$$= \frac{x}{4} - \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{8}\int \cos(w)\cdot\frac{1}{4}dw$$

$$= \frac{3x}{8} - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(w) + C$$

$$= \frac{3x}{8} - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

... it's not clear that this is better than using the reduction formulas.

For even powers of cos or sin $\geq 6$, the reduction formulas are likely the better choice.

$3^\circ$ $\int \sec^4(x)\,dx = \int \sec^2(x)\sec^2(x)\,dx$

This works for even powers of sec.

$$= \int (1+\tan^2(x))\sec^2(x)\,dx \qquad \begin{array}{l} u = \tan(x) \\ du = \sec^2(x)\,dx \end{array}$$

$$= \int (1+u^2)\,du = u + \frac{u^3}{3} + C$$

$$= \tan(x) + \frac{1}{3}\tan^3(x) + C$$

4°. $\boxed{\int \sec^3(x)\,dx} = \int \sec(x)\sec^2(x)\,dx = ??$

Use parts instead of substitution...

$u = \sec(x) \quad v' = \sec^2(x)$

$u' = \sec(x)\tan(x) \quad v = \tan(x)$

$= \sec(x)\tan(x) - \int \sec(x)\tan(x)\tan(x)\,dx$

$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$

$= \sec(x)\tan(x) - \int \sec(x)\left[\sec^2(x) - 1\right]dx$

$= \sec(x)\tan(x) - \boxed{\int \sec^3(x)\,dx} + \int \sec(x)\,dx$

$\Rightarrow 2\boxed{\int \sec^3(x)\,dx} = \sec(x)\tan(x) + \int \sec(x)\,dx$

All this is basically getting the reduction formula by using parts... more efficient to use the reduction formula to jump to

$\Rightarrow \int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx$

$= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln(\sec(x) + \tan(x)) + C$

5° $\int \tan^4(x)\,dx = \int \tan^2(x)\tan^2(x)\,dx$

$$= \int \tan^2(x)\left(\sec^2(x) - 1\right)dx$$

$$= \int \tan^2(x)\sec^2(x)\,dx - \int \tan^2(x)\,dx$$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$$= \int u^2\,du - \int \left(\sec^2(x) - 1\right)dx$$

$$= \frac{u^3}{3} - \int \sec^2(x)\,dx + \int 1\,dx$$

$$= \frac{u^3}{3} - \int 1\,du + x$$

$$= \frac{u^3}{3} - u + x + C$$

$$= \frac{1}{3}\tan^3(x) - \tan(x) + x + C$$

Again, this replicates the work needed to get the reduction formula for $\int \tan^n(x)\,dx$.

6° $\int \tan^5(x)\,dx = \int \tan^4(x)\tan(x)\,dx$

$$= \int \left(\tan^2(x)\right)^2 \tan(x)\,dx$$

$$= \int \left(\sec^2(x) - 1\right)^2 \tan(x)\,dx$$

$$= \int \left(\sec^4(x) - 2\sec^2(x) + 1\right)\tan(x)\,dx$$

$$= \int \sec^4(x)\tan(x)\,dx - 2\int \sec^2(x)\tan(x)\,dx + \int \tan(x)\,dx$$

$u = \sec(x) \quad du = \sec(x)\tan(x)\,dx$

$$= \int u^3\,du - 2\int u\,du + \ln(\sec(x))$$

$$= \frac{u^4}{4} - \cancel{2}\cdot\frac{u^2}{\cancel{2}} + \ln(\sec(x)) + C$$

$$= \frac{\sec^4(x)}{4} - \sec^2(x) + \ln(\sec(x)) + C$$

Again, this replicates the work that might go into the corresponding reduction formula for tan ...

Next time: mixes of trig functions in the integrand.