Integration by Parts II – a more detailed rule of thumb, more examples, and a side of hyperbolic functions

Note: Substitution is §8.1 in the text & Parts is §8.4. We'll go back and do §8.2 & §8.3 after §8.4 because the best formulas to help with trig integrals require parts.

$$\int u \cdot v' dx = u \cdot v - \int u' \cdot v \, dx$$

The more detailed rule of thumb for integration by parts:
With an integrand that is the product of two dissimilar functions, put whichever appears first on the list of types below into u:

1. logarithmic & inverse hyperbolic functions
2. inverse trigonometric
3. polynomials (& other powers of x)
4. trigonometric functions.
5. exponential & hyperbolic functions.

eg $\int x^2 \cos(x) dx$

$\Rightarrow$ put $u = x^2$

& $v' = \cos(x)$

<u>Aside:</u> The hyperbolic functions are defined in terms of exponential functions.

"kosh" $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

"sinch" $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

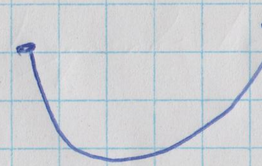
"tanch" $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

"seech" $\mathrm{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{2}{e^x + e^{-x}}$

Their inverses are basically logarithmic,

These arise in solving various differential equations

Es hang a chain from two points it makes a "catenary" curve, which is a scaled version of $\cosh(x)$.



~~and~~ as alternatives to the trig functions in doing substitutions.

Examples: 1° This one was left ~~unf~~ unfinished when we did substitution...

$$\int x^3 e^{x^2}\,dx$$

Applying the detailed rule of thumb doesn't quite work:

$u = x^3 \qquad v' = e^{x^2}$

$\downarrow \qquad\qquad \downarrow$

$u' = 3x^2 \qquad ?$

$\boxed{\text{Fact: } e^{x^2},\ e^{-x^2} \text{ \& others have no nice antiderivatives}}$

$$= \int \underbrace{x \cdot x^2 \cdot e^{x^2}\,dx}$$

so we substitute $w = x^2$ to simplify first:

Then $dw = 2x\,dx$, so $x\,dx = \frac{1}{2}\,dw$, so

$$= \frac{1}{2}\int w e^w\,dw$$

Now the rule of thumb suggests

$u = w$ & $v' = e^w$, so

$u' = 1$ & $v = e^w$.

$$= \frac{1}{2}\left[w e^w - \int 1\cdot e^w\,dw\right] = \frac{1}{2}\left[w e^w - e^w\right] + C$$

$$= \frac{e^w}{2}(w-1) + C = \boxed{\frac{e^{x^2}}{2}(x^2-1) + C}$$

$2^\circ \quad \int_0^1 \overset{u\cdot v'}{x^2 e^x}\,dx$

Using the rule of thumb,

$u = x^2$ & $v' = e^x$

& $u' = 2x$ & $v = e^x$.

$$= \overset{u\cdot v}{x^2 e^x}\Big|_0^1 - \int_0^1 \overset{u'\cdot v}{\overset{s\cdot t'}{2xe^x}}\,dx$$

Also a product of two dissimilar functions, so we use parts & the rule of thumb:

$s = 2x$ & $t' = e^x$, so

$s' = 2$ & $t = e^x$

$$= \left(1^2\cdot e^1 - 0^2\cdot e^0\right) - \left[\overset{s\cdot t}{2xe^x}\Big|_0^1 - \int_0^1 \overset{s'\cdot t}{2e^x}\,dx\right]$$

$$= (e-0) - \left[\left(2\cdot 1\cdot e^1 - 2\cdot 0\cdot e^0\right) - 2e^x\Big|_0^1\right]$$

$$= e - \left[(2e-0) - \left(2\cdot e^1 - 2\cdot e^0\right)\right]$$

$$= e - \left[\cancel{2e} - \left(\cancel{2e} - 2\cdot 1\right)\right] = \boxed{e-1}$$

Moral: you might need to use parts more than once...

3° $\boxed{\int e^x \cos(x)\,dx}$ Following the "rule", put

$u = \cos(x)$ & $v' = e^x$,

so $u' = -\sin(x)$ & $v = e^x$

$$= e^x\cos(x) - \int e^x(-\sin(x))\,dx$$

$$= e^x\cos(x) + \int e^x \sin(x)\,dx$$

Follow the rule of thumb:

$s = \sin(x)$ & $t' = e^x$,

so $s' = \cos(x)$ & $t = e^x$.

$$= e^x\cos(x) + \left[e^x\sin(x) - \int e^x\cos(x)\,dx \right]$$

$$= e^x\cos(x) + e^x\sin(x) - \boxed{\int e^x\cos(x)\,dx}$$

Since we started with $\int e^x\cos(x)\,dx$, now what? Solve for what's in the box $\boxed{\ldots}$

$$2\boxed{\int e^x\cos(x)\,dx} = e^x\cos(x) + e^x\sin(x), \text{ \& so}$$

$$\boxed{\int e^x\cos(x)\,dx} = \frac{1}{2}\left(e^x\cos(x) + e^x\sin(x)\right) + C$$