Sequences (and their limits) (Section 11.1)

A sequence is a list of numbers, indexed by the integers (from some point on).

eg $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Written as $\left\{\frac{1}{2^n}\right\}$ or $\left\{\frac{1}{2^n} \mid n \geq 0\right\}$ or $\left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$ or ...

or just $\frac{1}{2^n}$.

If we're talking about sequences in general we usually use $\{a_n\}$ or a_n.

Defn: The limit of a sequence a_n is L

ie $\lim_{n\to\infty} a_n = L$, if

For all $\varepsilon > 0$, there is an N s.t. for all $n \geq N$,

$$|a_n - L| < \varepsilon.$$

We don't usually use this definition but rely on computational rules that can be proved using this definition.

Example of using the definition:

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

Suppose $\varepsilon > 0$ is given. We need to find an N s.t. if $n \geq N$,

then $\left|\frac{1}{2^n} - 0\right| < \varepsilon$.

Reverse-engineer the required N from the ε:

$$\left|\frac{1}{2^n} - 0\right| < \varepsilon$$

$$\iff \frac{1}{2^n} < \varepsilon \iff \frac{1}{\varepsilon} < 2^n$$

$$\iff n > \log_2\left(\frac{1}{\varepsilon}\right)$$

Pick any integer N s.t. $N > \log_2\left(\frac{1}{\varepsilon}\right)$.

Then if $n \geq N$, we have

$$n \geq N > \log_2\left(\frac{1}{\varepsilon}\right)$$

(& following our derivation backwards)

$$\Rightarrow \left|\frac{1}{2^n} - 0\right| < \varepsilon.$$

$\therefore \lim_{n\to\infty} \frac{1}{2^n} = 0$ by def'n.

In practice we rely on various computational properties of limits and also some basic theorems.

1. $\lim\limits_{n\to\infty}(a_n+b_n) = \left(\lim\limits_{n\to\infty} a_n\right) + \left(\lim\limits_{n\to\infty} b_n\right)$
   (provided that all these limits exist)

2. $\lim\limits_{n\to\infty}(c a_n) = c\cdot\left(\lim\limits_{n\to\infty} a_n\right)$

3. $\lim\limits_{n\to\infty} a_n\cdot b_n = \left(\lim\limits_{n\to\infty} a_n\right)\cdot\left(\lim\limits_{n\to\infty} b_n\right)$

4. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\left(\lim\limits_{n\to\infty} a_n\right)}{\left(\lim\limits_{n\to\infty} b_n\right)}$ provided that also $\lim\limits_{n\to\infty} b_n \neq 0$

& so on.

5. If $a_n = f(n)$ for some continuous function $f(x)$
   then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} f(n) = \lim\limits_{x\to\infty} f(x)$
   (provided the last one exists).

   <u>Warning</u>: $\lim\limits_{x\to\infty} f(x)$ might not exist even if $\lim\limits_{n\to\infty} a_n$ does exist.

   <u>eg</u> $f(x) = \sin(\pi x)$ has no limit as $x\to\infty$

   but $\lim\limits_{n\to\infty} \sin(\pi\cdot n) = \lim\limits_{n\to\infty} 0 = 0$.

<u>Note:</u> We can't use l'Hôpital's Rule directly to compute limits of <u>sequences</u>, but we can use via rule 5 by converting to a continuous variable like x – provided it is applicable there!

<u>eg</u> $a_n = \frac{n}{e^n} \quad n \geq 0$

$0, \frac{1}{e}, \frac{2}{e^2}, \frac{3}{e^3}, \ldots$

$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{x\to\infty} \frac{x \to \infty}{e^x \to \infty}$ so apply l'Hôpital's Rule

$= \lim_{x\to\infty} \frac{\frac{d}{dx} x}{\frac{d}{dx} e^x} = \lim_{x\to\infty} \frac{1 \to 1}{e^x \to \infty} = 0$

We can't use this trick all the time because the sequence might not be easily definable in terms of a continuous variable.

<u>eg</u> $n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$
$n \geq 1$

n	1	2	3	4	5	$\rightarrow$
$n!$	1	2	6	24	120	$\rightarrow$

($\Gamma(x)$ is a function of x s.t. $n! = \Gamma(n+1)$ but $\Gamma(x)$ is hard to work with.)

6 Squeeze Theorem

If $a_n \le b_n \le c_n$ for all n past some point, then if you also have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n$, we must have $\lim\limits_{n\to\infty} b_n$ exist and be equal to the other two.

eg $$\lim_{n\to\infty} \frac{2^n}{n!} = 0$$

$$0 \le \frac{2^n}{n!} = \frac{2\cdot 2\cdot 2\cdot 2 \cdots 2\cdot 2}{\underbrace{n\cdot(n-1)(n-2)(n-3)\cdots 2\cdot 1}_{\le 1}} \le \frac{2}{n}$$

as $n\to\infty$: $0 \to 0$ and $\frac{2}{n} \to 0$

$\therefore$ $\lim\limits_{n\to\infty} \frac{2^n}{n!} = 0$ too by the Squeeze Theorem.

7 Monotonic Convergence Theorem

Suppose a_n is a non-decreasing sequence with an upper bound (or a non-increasing sequence with a lower bound). Then $\lim_{n\to\infty} a_n$ exists.

[The limit is the least upper bound (or greatest lower bound) of the sequence]

eg $a_n = \frac{1}{2^n}$ is decreasing

because $0 < \frac{1}{2^{n+1}} = \frac{1}{2} \cdot \frac{1}{2^n} < \frac{1}{2^n}$

& has 0 as a lower bound

so $\lim_{n\to\infty} \frac{1}{2^n}$ exists by the Monotone Convergence Theorem.