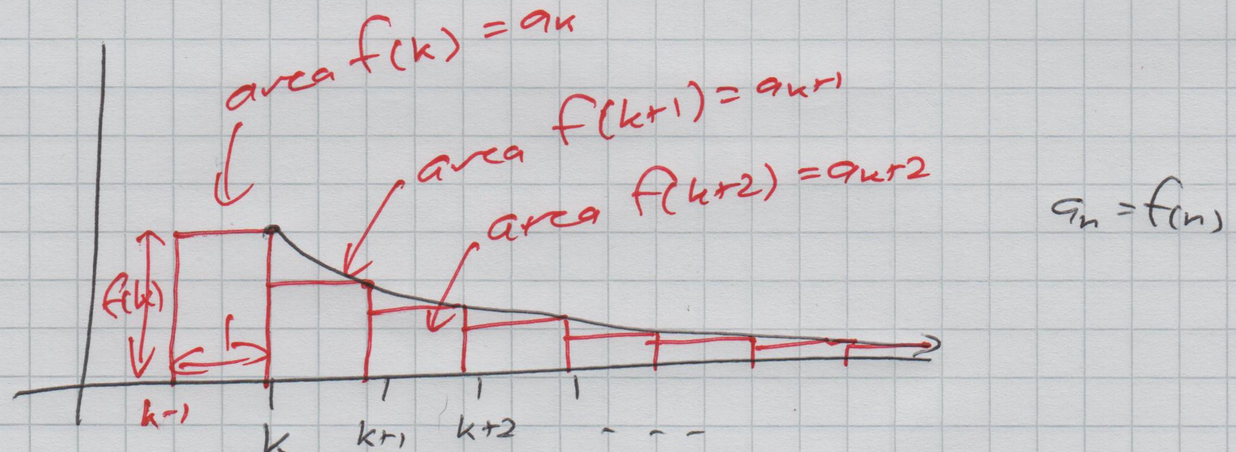
Integral Test and the p-Tests (§11.3 in the text)

Integral Test Suppose we have a function $f(x) > 0$ defined and decreasing (and integrable) on an interval $[k, \infty)$ for some integer k. If $a_n = f(n)$, then the series $\sum_{n=k}^{\infty} a_n$ converges exactly when the integral $\int_k^{\infty} f(x)\,dx$ converges (i.e. works out to a real number).

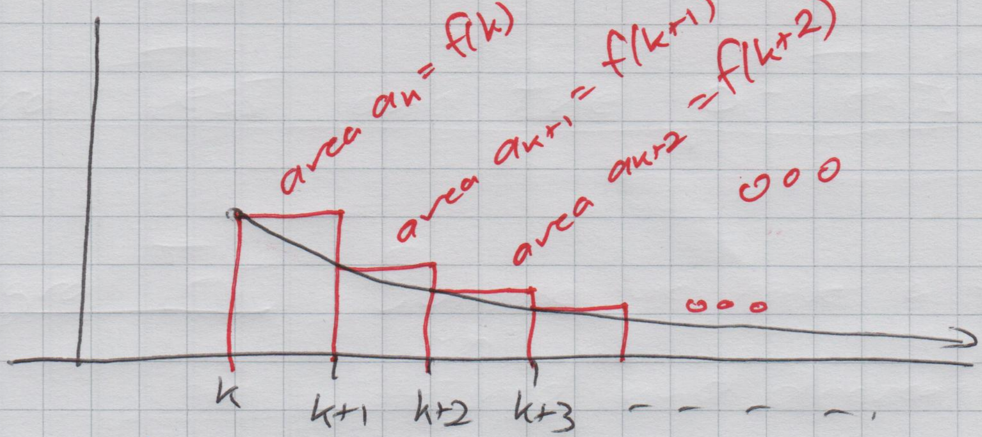
Why does this work?



So if $\int_k^{\infty} f(x)\,dx$ converges, the sum

$$a_k + \sum_{n=k+1}^{\infty} a_n \leq a_k + \int_k^{\infty} f(x)\,dx$$

so the sum converges by the Monotonic Convergence thm



If $\int_k^\infty f(x)\,dx$ diverges – which means the area under the graph is infinite, then the sum of the areas of the rectangles, $\sum_{n=k}^\infty a_n$, must be infinite also, i.e. $\sum_{n=k}^\infty a_n$ diverges.

eg Harmonic series diverges by the Integral Test:

$\sum_{n=1}^\infty \frac{1}{n}$ this converges (or not) as $\int_1^\infty \frac{1}{x}\,dx$

($f(x) > 0$ & decreasing & continuous on $[1, \infty)$)

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty} \ln(x)\Big|_1^a$$

$$= \lim_{a\to\infty} \left[\underset{\downarrow\atop\infty}{\ln(a)} - \underset{0}{\ln(1)}\right] = \infty$$

$\therefore \sum_{n=1}^\infty \frac{1}{n}$ diverges by the Integral Test.

eg the "Square-harmonic" series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

converges by the Integral Test:

$f(x) = \frac{1}{x^2} > 0$, & decreasing, & continuous (& so integrable)

on $[1, \infty)$. (And $f(n) = \frac{1}{n^2}$.)

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{a \to \infty} \int_1^a \frac{1}{x^2}\,dx = \lim_{a \to \infty} \int_1^{a} x^{-2}\,dx$$

$$= \lim_{a \to \infty} \frac{x^{-2+1}}{-2+1}\Big|_1^a = \lim_{a \to \infty} -x^{-1}\Big|_1^a$$

$$= \lim_{a \to \infty} \frac{-1}{x}\Big|_1^a = \lim_{a \to \infty} \left(\underset{\downarrow\, 0}{\frac{-1}{a}} + \frac{+1}{1}\right) = -0 + 1 = 1 \in \mathbb{R}$$

so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the Integral Test.

<u>Notice</u>: This gives very little information on what the series converges to.

(It actually converges to $\frac{\pi^2}{6}$.)

The <u>p-Test</u> is:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges exactly when } p>1.$$

(& so it diverges if $p \le 1$.).

The <u>General p-Test</u> is:

$$\sum_{n=1}^{\infty} a_n \quad \text{where} \quad a_n = \frac{b_k n^k + b_{k-1} n^{k-1} + \cdots + b_1 n + b_0}{c_\ell n^\ell + c_{\ell-1} n^{\ell-1} + \cdots + c_1 n + c_0},$$

then this will converge when $p = \ell - k > 1$

& diverge when $p = \ell - k \le 1$.

(The proof of the p-Test is basically applying the Integral Test. The proof of the General p-Test is usually done by combining the p-Test with Limit Comparison Test (coming later!)).