Applications I (Covering parts of § 9.1, 9.3, 9.4, 9.9)

[average values and geometrical quantities]

1. Average value of a function (§9.4)

Average value of $f(x)$ on $[a,b]$

is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

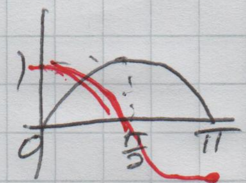
analogue of $\left[\frac{a_1+\cdots+a_n}{n}\right]$

eg $f(x)=1$ and $[1,3]$

$$\text{Average value} = \frac{1}{3-1}\int_1^3 1\,dx = \frac{1}{2}x\Big|_1^3$$

$$= \frac{1}{2}\cdot 3 - \frac{1}{2}\cdot 1 = \frac{1}{2}(3-1) = \frac{1}{2}\cdot 2 = 1$$

eg $f(x) = \sin(x)$ on $[0,\pi]$



$$\text{Average value} = \frac{1}{\pi - 0}\int_0^\pi \sin(x)\,dx$$

$$= \frac{1}{\pi}(-\cos(x))\Big|_0^\pi = \left[\frac{-1}{\pi}\cos(\pi)\right] - \left[\frac{-1}{\pi}\cos(0)\right]$$

$$= \left[\frac{-1}{\pi}(-1)\right] - \left[\frac{-1}{\pi}\cdot 1\right] = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

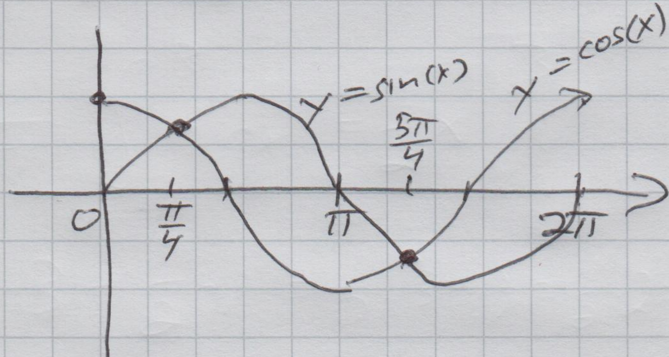
2. Areas between curves (§9.1)

Area of the region between $y = f(x)$ and $y = g(x)$ for $a \le x \le b$ is

$$\int_a^b (\text{upper} - \text{lower})\,dx$$

Need to figure which function is upper and which is lower and break up the integral accordingly.

eg Area between $y = \sin(x)$ & $y = \cos(x)$ for $0 \le x \le 2\pi$.



Between 0 & $\frac{\pi}{4}$ $\cos(x)$ is above $\sin(x)$.

Between $\frac{\pi}{4}$ & $\frac{5\pi}{4}$ $\sin(x)$ is above $\cos(x)$.

Between $\frac{5\pi}{4}$ and 2π, $\cos(x)$ is above $\sin(x)$

$$\text{Area} = \int_0^{2\pi} (\text{upper} - \text{lower})\,dx$$

$$= \int_0^{\pi/4} (\cos(x) - \sin(x))\,dx + \int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\,dx$$

$$+ \int_{5\pi/4}^{2\pi} (\cos(x) - \sin(x))\,dx$$

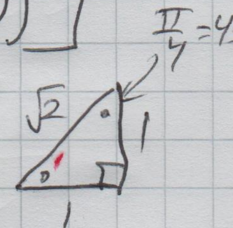
$$= \left[\sin(x) + (+\cos(x))\right]\Big|_0^{\pi/4} + \left[-\cos(x) - \sin(x)\right]\Big|_{\pi/4}^{5\pi/4}$$

$$+ \left[\sin(x) + (+\cos(x))\right]\Big|_{5\pi/4}^{2\pi}$$

$$= \left[\left(\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)\right) - \left(\sin(0) + \cos(0)\right)\right]$$

$$+ \left[\left(-\cos\left(\tfrac{5\pi}{4}\right) - \sin\left(\tfrac{5\pi}{4}\right)\right) - \left(-\cos\left(\tfrac{\pi}{4}\right) - \sin\left(\tfrac{\pi}{4}\right)\right)\right]$$

$$+ \left[\left(\sin(2\pi) + \cos(2\pi)\right) - \left(\sin\left(\tfrac{5\pi}{4}\right) + \cos\left(\tfrac{5\pi}{4}\right)\right)\right]$$



$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right]$$

$$+ \left[\left(-\left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}}\right)\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$$

$$+ \left[(0+1) - \left(\frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\right)\right]$$

so $\cos\left(\frac{\pi}{4}\right)$
$= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
& $\cos\left(\frac{5\pi}{4}\right)$
$= \sin\left(\frac{5\pi}{4}\right) = \frac{-1}{\sqrt{2}}$

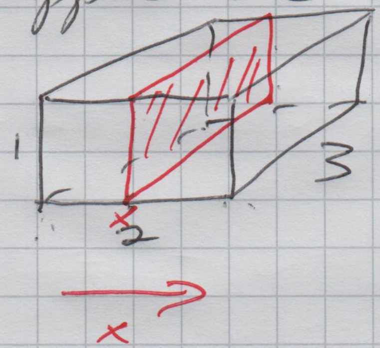
$$= \left[\frac{2}{\sqrt{2}} \cancel{-1}\right] + \left[\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right] + \left[\cancel{1+} \frac{2}{\sqrt{2}}\right]$$

$$= \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

3. Volumes (§9.3)

Basic idea: area of a cross-section sweeps out volume

eg Suppose we have a rectangular box

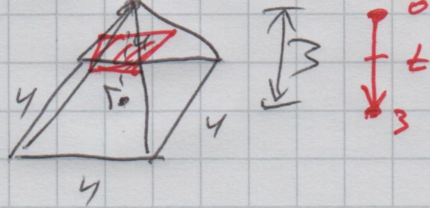


$V = 1 \cdot 2 \cdot 3 = 6$

$A(x) = 1 \cdot 3 = 3 \qquad 0 \le x \le 2$

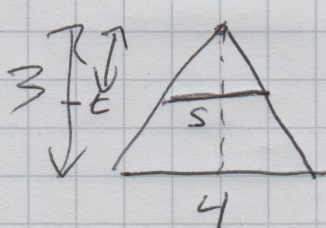
$$V = \int_0^2 A(x)\,dx = \int_0^2 3\,dx = 3x\Big|_0^2 = 3\cdot 2 - 3\cdot 0 = 6 \checkmark$$

eg Suppose we have a square-based pyramid (with base side length $= 4$) and height 3.



The cross section at t = vertical distance from the tip

$A(t) = ?$ $= (\text{sidelength})^2$



$t : s$ as $3 : 4$ by similar triangles

$\frac{s}{t} = \frac{4}{3} \Rightarrow s = \frac{4}{3}t$

$A(t) = s^2 = \frac{16}{9}t^2$

$$V = \int_0^3 A(t)\,dt = \int_0^3 \frac{16}{9} t^2\,dt$$

$$= \frac{16}{9} \cdot \frac{t^3}{\cancel{27}\,3} \Big|_0^3 = \frac{16}{9} \cdot \frac{3^3}{\cancel{27}\,3} - \cancel{\frac{16}{9} \cdot \frac{0^3}{\cancel{27}\,3}} \qquad 3^3 = 27$$

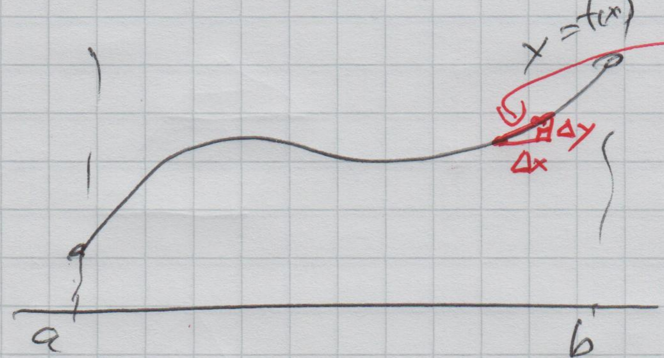
$$= \cancel{\frac{16}{9}}\ \frac{16}{\cancel{9}} \cdot \frac{\cancel{3^3}}{\cancel{3}} = 16 \text{ units}^3$$

Solids of revolution will be covered in Applications II.

4. Arc-lengths of curves (§9.9)

Warning: Usually give hard integrals.

Problem: How long is the curve $y = f(x)$ for $a \leq x \leq b$?



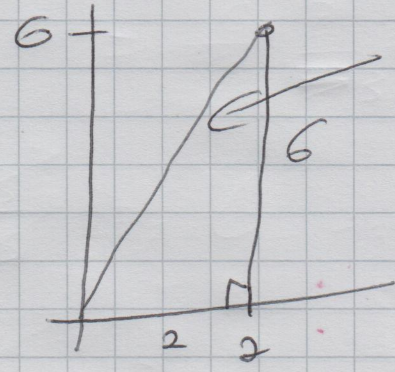
length $= \sqrt{(\Delta y)^2 + (\Delta x)^2}$

$$= \sqrt{\left[\left(\frac{\Delta y}{\Delta x}\right)^2 + 1\right](\Delta x)^2}$$

$$= \sqrt{\left(\frac{\Delta y}{\Delta x}\right)^2 + 1}\ \Delta x$$

As we shrink Δx & Δy, this comes down to $ds = \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\ dx$ = infinitesimal increment of arc-length.

es $\quad y = 3x \qquad 0 \le x \le 2$



$$\text{length} = \sqrt{2^2+6^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$$

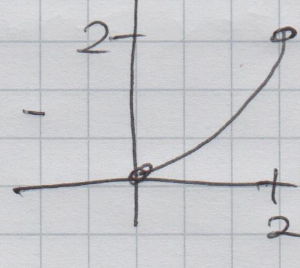
$$\frac{dy}{dx} = 3$$

$$\text{Arc-length} = \int_0^2 ds = \int_0^2 \sqrt{\left(\frac{dy}{dx}\right)^2+1}\, dx$$

$$= \int_0^2 \sqrt{3^2+1}\, dx = \int_0^2 \sqrt{10}\, dx = \sqrt{10}\cdot x \Big|_0^2$$

$$= \sqrt{10}\cdot 2 - \sqrt{10}\cdot 0 = 2\sqrt{10} \quad \checkmark$$

es $\quad y = \dfrac{x^2}{2} \qquad 0 \le x \le 2$



$$\frac{dy}{dx} = \frac{2x}{2} = x$$

$$\text{Arc-length} = \int_0^2 ds = \int_0^2 \sqrt{x^2+1}\, dx \qquad \begin{array}{l} x = \tan(t) \\ dx = \sec^2(t) \end{array}$$

$$= \int_{x=0}^{x=2} \sqrt{\underbrace{\tan^2(t)+1}_{\sec^2(t)}}\cdot \sec^2(t)\, dt = \int_{x=0}^{x=2} \sec^3(t)\, dt$$

$$= \frac{1}{3-1} \tan(t)\sec^{3-2}(t) \Big|_{x=0}^{x=2} + \frac{3-2}{3-1} \int_{x=0}^{x=2} \sec^{3-2}(x)\, dx$$

$$= \frac{1}{2} \tan(t)\sec(t) \Big|_{x=0}^{x=2} + \frac{1}{2} \int_{x=0}^{x=2} \sec(x)\, dx$$

$$= \frac{1}{2} \tan(t)\sec(t) \Big|_{x=0}^{x=2} + \frac{1}{2} \ln(\sec(t) + \tan(x)) \Big|_{x=0}^{x=2}$$

$$x = \tan(t) \qquad \sec(t) = \sqrt{x^2+1}$$

$$= \frac{1}{2} x \sqrt{x^2+1} \Big|_0^2 + \frac{1}{2} \ln\left(x + \sqrt{x^2+1}\right) \Big|_0^2$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2} \sqrt{2^2+1} - \cancel{\frac{1}{2} \cdot 0 \sqrt{0^2+1}}$$

$$+ \frac{1}{2} \ln\left(2 + \sqrt{2^2+1}\right) - \frac{1}{2} \ln\left(0 + \sqrt{0^2+1}\right)$$

$$= \boxed{\sqrt{5} + \frac{1}{2} \ln\left(2 + \sqrt{5}\right)} - \cancel{\frac{1}{2} \ln(1)}$$