Alternating Series Test (§11.5)

and absolute vs. conditional convergence (§11.6)

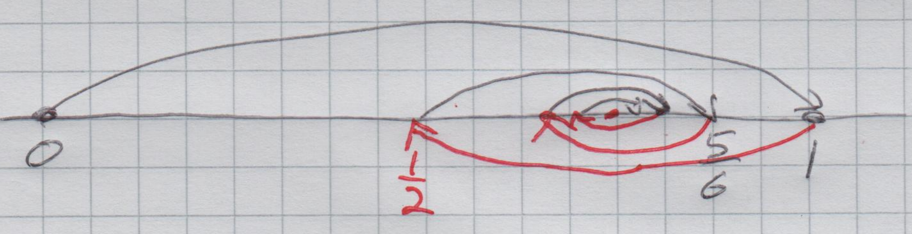
An alternating series is a series in which the term alternate sign, i.e. a negative term is always followed by a positive term & vice versa.

The prototype is the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Unlike the harmonic series, this converges.

Why?



Alternating Series Test

Suppose $\sum_{n=0}^{\infty} a_n$ is a series. Then, if

(0) the series alternates

and (1) past some point, $|a_{n+1}| < |a_n|$ for all n

and (2) $\lim_{n\to\infty} |a_n| = 0$,

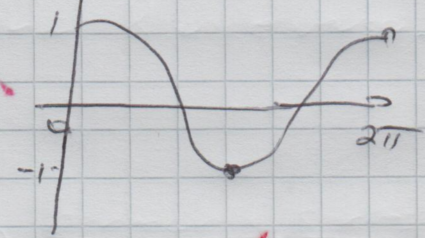
the series converges.

Example: $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n[\ln(n)]^2}$

(0) Does the series alternate?

$n[\ln(n)]^2 > 0 \quad \text{for } n \geq 2$

n	$\cos(n\pi) = (-1)^n$
2	1
3	-1
4	1
5	-1
⋮	⋮



Yes, it alternates...

(1) Do we have $|a_{n+1}| < |a_n|$ past some point?

$\ln(n)$ is an increasing function, as is n

so $n[\ln(n)]^2$ is increasing, so

$$(n+1)[\ln(n+1)]^2 > n[\ln(n)]^2,$$

so $$\underbrace{\frac{1}{(n+1)[\ln(n+1)]^2}}_{=|a_{n+1}|} < \underbrace{\frac{1}{n[\ln(n)]^2}}_{=|a_n|}$$

Yes, we do, for all $n \geq 2$.

(2) Do we have $\lim_{n\to\infty} |a_n| = 0$?

$$\lim_{n\to\infty} \left| \frac{\cos(n\pi)}{n[\ln(n)]^2} \right| = \lim_{n\to\infty} \frac{1 \to 1}{n[\ln(n)]^2 \to \infty} = 0$$

because $n \to \infty$
& $\ln(n) \to \infty$
& $[\ln(n)]^2 \to \infty$
& $n[\ln(n)]^2 \to \infty$

as $n \to \infty$

Yes, we do.

Since we have satisfied all three conditions for the Alternating Series Test,

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n[\ln(n)]^2} \quad \text{converges.}$$

Q.: Does $\displaystyle\sum_{n=2}^{\infty} \left| \frac{\cos(n\pi)}{n[\ln(n)]^2} \right|$ converge?

Yes: $\displaystyle\sum_{n=2}^{\infty} \left| \frac{\cos(n\pi)}{n[\ln(n)]^2} \right| = \sum_{n=2}^{\infty} \frac{1}{n[\ln(n)]^2}$, and this

converges by the Integral Test.

$$\int_2^{\infty} \frac{1}{x[\ln(x)]^2}\,dx$$ Try substituting $u = \ln(x)$, so $du = \frac{1}{x}\,dx$

x	u
2	$\ln(2)$
∞	∞

$$= \int_{\ln(2)}^{\infty} \frac{1}{u^2}\,du$$

$$= \int_{\ln(2)}^{\infty} u^{-2}\,du = \frac{u^{-2+1}}{-2+1}\Big|_{\ln(2)}^{\infty} = \frac{u^{-1}}{-1}\Big|_{\ln(2)}^{\infty} = \frac{-1}{u}\Big|_{\ln(2)}^{\infty}$$

$$= \left(\frac{-1}{\infty}\right) - \left(\frac{-1}{\ln(2)}\right) = 0 + \frac{1}{\ln(2)}$$ [We get away with tossing ∞ about like a number...]

Since this improper integral converges,

$$\sum_{n=2}^{\infty} \frac{1}{n[\ln(n)]^2} \text{ converges,}$$

so $$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n[\ln(n)]^2}$$ "converges absolutely".

<u>Defn:</u> If a series $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ <u>converges absolutely</u>.

If $\sum_{n=0}^{\infty} |a_n|$ diverges, but $\sum_{n=0}^{\infty} a_n$ converges, it is said to <u>converge conditionally</u>.

Facts: ① No matter how rearrange the terms in an absolutely convergen series, you will get the same sum.

② You can rearrange the terms of a conditionally convergent series to get any sum you want (or to get it to diverge).

Example: We'll make the alternating harmonic series [which is conditionally convergent] to equal, say, 1.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

Note that $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$

and $-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \cdots$

both diverge

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{6} + \cdots = 1$$

$=1$ <1 >1 <1 >1