Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2020

Solutions to Assignment #5
A Little Series Calculus

1. Find a nice formula for the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

and determine for which values of x this series converges. *[1]*

Solution. Observe that $\dfrac{(-1)^{n+1}x^{n+1}}{(-1)^n x^n} = -x$ for all $n \geq 0$, so successive terms of this series always have a ratio of $-x$. It follows that this series is a geometric series with first term $a = 1$ and common ratio $r = -x$, and so it converges when $|r| = |-x| = |x| < 1$ and diverges otherwise. Moreover, when it converges, it converges to $\dfrac{a}{1-r} = \dfrac{1}{1-(-x)} = \dfrac{1}{1+x}$. $\square$

2. Find a nice formula for the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n n x^{n-1} = 0 - 1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots$$

and determine for which values of x the series converges. *[2]*

Solution. Recall the Power Rule for differentiation: $\dfrac{d}{dx}x^n = nx^{n-1}$. With this rule in mind it is pretty obvious that the series in this question is the derivative of the series in question **1**:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n x^n\right) = \sum_{n=0}^{\infty}(-1)^n \frac{d}{dx}x^n = \sum_{n=0}^{\infty}(-1)^n n x^{n-1}$$

Since the geometric series series in question **1** equals $\dfrac{1}{1+x} = (1+x)^{-1}$ when it converges, its derivative would have to equal

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}(1+x)^{-1} = (-1)(1+x)^{-2} \cdot \frac{d}{dx}(1+x) = (-1)(1+x)^{-2} \cdot 1 = \frac{-1}{(1+x)^2}$$

when it converges, *i.e.* $\displaystyle\sum_{n=0}^{\infty}(-1)^n n x^{n-1} = \frac{-1}{(1+x)^2}$.

It remains to determine when the given series converges. We first try the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)x^{(n+1)-1}}{(-1)^n n x^{n-1}}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)x^n}{(-1)^n n x^{n-1}}\right| = \lim_{n\to\infty}\left|(-1)\frac{n+1}{n}x\right| \\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)|x| = |x| \quad \text{as } \frac{1}{n} \to 0 \text{ as } n \to \infty.\end{aligned}$$

It follows by the Ratio Test that the series converges when $|x| < 1$ and diverges when $|x| > 1$. However, the Ratio Test tells us nothing when $|x| = 1$, *i.e.* when $x = \pm 1$, so we handle these cases separately.

When $x = -1$, the given series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n n (-1)^{n-1} = \sum_{n=0}^{\infty} (-1)^{2n-1} n = \sum_{n=0}^{\infty} (-n)$. This diverges by the Divergence Test because $\displaystyle\lim_{n\to\infty} |-n| = \lim_{n\to\infty} n = \infty \neq 0$.

Similarly, when $x = 1$ the given series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n n 1^{n-1} = \sum_{n=0}^{\infty} (-1)^n n$. This also diverges by the Divergence Test because $\displaystyle\lim_{n\to\infty} |(-1)^n n| = \lim_{n\to\infty} n = \infty \neq 0$.

Thus the given series sums to $\frac{-1}{(1+x)^2}$ when it converges, and it converges for $|x| < 1$ and diverges otherwise. $\square$

3. Find a nice formula for the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

and determine for which values of x this series converges. *[3]*

Solution. Recall the Power Rule for integration: $\displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1}$ (up to a constant), unless $n = -1$. With this rule in mind it is pretty obvious that the series in this question is the antiderivative of the series in question **1**:

$$\int \left(\sum_{n=0}^{\infty} (-1)^n x^n \right) dx = \sum_{n=0}^{\infty} (-1)^n \int x^n \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

As the given series has 0 for a constant term, we take $C = 0$.

Since the geometric series series in question **1** equals $\dfrac{1}{1+x} = (1+x)^{-1}$ when it converges, its antiderivative would have to equal

$$\int \frac{1}{1+x} \, dx = \int \frac{1}{u} \, du = \ln(u) + B = \ln(1+x) + B$$

for some constant B. We can determine B by plugging in $x = 0$ into both $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ and $B + \ln(1+x)$. As $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 0^{n+1}}{n+1} = 0 = \ln(1+0)$, it follows that $B = 0$. Thus the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ sums to $\ln(1+x)$ when it converges.

It remains to determine when the given series converges. We try the Ratio Test again:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{(n+1)+1}}{(n+1)+1}}{\frac{(-1)^n x^{n+1}}{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+2}}{n+2}\cdot\frac{n+1}{(-1)^n x^{n+1}}\right| = \lim_{n\to\infty}\left|(-1)x\frac{n+1}{n+2}\right|$$

$$= |x|\lim_{n\to\infty}\frac{n+1}{n+2}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = |x|\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}} = |x|\frac{1+0}{1+0} = |x|$$

as $\frac{1}{n}\to 0$ and $\frac{2}{n}\to 0$ as $n\to\infty$. It follows by the Ratio Test that the series converges when $|x|<1$ and diverges when $|x|>1$. However, the Ratio Test tells us nothing when $|x|=1$, *i.e.* when $x=\pm 1$, so we handle these cases separately.

When $x=-1$ the given series is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{-1}{n+1}$, *i.e.* the negative of the harmonic series. This diverges by the Generalized p-Test because the degree of $-1=-1\cdot n^0$ is 0 and the degree of $n=n^1$ is 1, so for this series $p=1-0=1\leq 1$.

Similarly, when $x=1$ the given series is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 1^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 1-\frac{1}{2}+\frac{1}{3}-\cdots$, *i.e.* the alternating harmonic series. This converges by the Alternating Series Test:

i. Since $(-1)^n$ alternates between 1 and -1 and $n+1>0$ for all $n\geq 0$, $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ is indeed an alternating series.

ii. Since $n+2>n+1$ for all n, we have $\displaystyle\left|\frac{(-1)^{n+1}}{(n+1)+2}\right| = \frac{1}{n+2} < \frac{1}{n+1} = \left|\frac{(-1)^n}{n+1}\right|$ for all n.

iii. $\displaystyle\lim_{n\to\infty}\left|\frac{(-1)^n}{n+1}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0$ since $n+1\to\infty$ as $n\to\infty$.

Since it satisfies all three hypothese of the Alternating Series Test, the given series converges when $x=1$. (Note that since the harmonic series diverges, the alternating harmonic series only converges conditionally.)

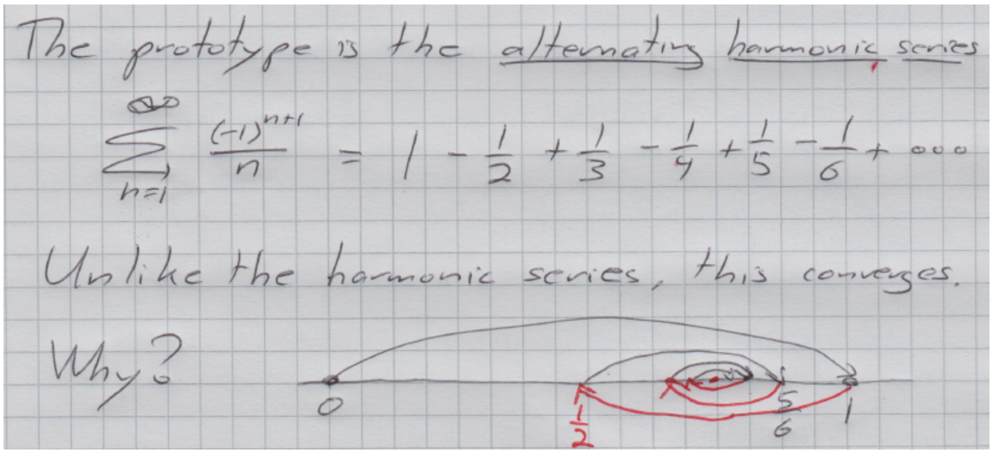
It follows that the given series converges when $-1<x\leq 1$ and diverges otherwise. When it converges, it converges to $\ln(1+x)$. $\square$

4. Find the sum of the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. How many terms of this series do you need to add up to guarantee that the partial sum in question is within $0.0001 = 10^{-4} = \frac{1}{10000}$ of the sum of the entire series? *[4]*

Solution. As noted above, we get the alternating harmonic series when we plug $x=1$ into the power series given in question **3**. This means that the sum of the harmonic series is whatever we get when we plug $x=1$ into the sum of the power series in **3**, $\ln(1+x)$. It follows that the sum of the alternating harmonic series is $\ln(1+1)=\ln(2)$.

How do we determine how many terms of the series we need to add up to ensure that the partial sum in question is within 0.0001 of $\ln(2)$? Recall the reason why the Alternating

Series Test works, pictured for the alternating harmonic series in Figure 11.4.1 of §11.4 in the textbook, and in the similar diagram from our lecture about the alternating series test:



Because the series alternates sign while the terms decrease in absolute value, each partial sum that is above (respectively, below) the sum of the entire series is followed by a partial sum that is below (respectively, above) the sum of the entire series. This means that each partial sum differs from the sum of the entire series by less than the absolute value of the next term to be added or subtracted. That is:

$$\left| \left(\sum_{n=1}^{k} \frac{(-1)^{n+1}}{n} \right) - \ln(2) \right| < \left| \frac{(-1)^{k+2}}{k+1} \right| = \frac{1}{k+1}$$

To ensure that $\sum_{n=1}^{k} \frac{(-1)^{n+1}}{n}$ is within $0.0001 = \frac{1}{10000}$ of $\ln(2)$ it therefore suffices to ensure that $\frac{1}{k+1} \leq \frac{1}{10000}$. This will happen exactly when $k+1 \geq 10000$; that is, exactly when $k \geq 9999$.

It follows that if we add up the first 9999(or more) terms of the alternating harmonic series, we will be guaranteed that this sum is within 0.0001 of the sum of the entire series, $\ln(2)$. ■